ON THE ESSENTIAL SELF-ADJOINTNESS OF SUB-LAPLACIANS

VALENTINA FRANCESCHI[†], DARIO PRANDI[♭], AND LUCA RIZZI[♯]

Abstract. We prove a general essential self-adjointness criterion for sub-Laplacians on complete sub-Riemannian manifolds, defined with respect to singular measures. As a consequence, we show that the intrinsic sub-Laplacian (i.e. defined w.r.t. Popp's measure) is essentially self-adjoint on the equiregular connected components of a sub-Riemannian manifold. This result holds under mild regularity assumptions of the singular region, and when the latter does not contain characteristic points.

1. Introduction

It is well known that geometric singularities of a Riemannian structure can act as barriers for heat diffusion, wave propagation, and the evolution of quantum particles. Most surprisingly, this occurs even when the underlying Riemannian structure is not complete, and classical particles, whose trajectories are described by geodesics, can escape from the manifold in finite time. One of the simplest cases where this behavior can be observed is the *Grushin structure* given by the singular metric

$$g = dx \otimes dx + \frac{1}{x^2} dy \otimes dy. \tag{1}$$

This Riemannian structure on $\mathbb{R}^2 \setminus \{x = 0\}$ is clearly not geodesically complete, as almost all geodesics cross the singular region $\mathcal{Z} = \{x = 0\}$ in finite time. Moreover, the associated Riemannian volume $\frac{1}{|x|} dx \wedge dy$ explodes on $\mathcal{Z}$ and hence the corresponding Laplace-Beltrami operator presents both a degeneration and a singular drift on $\mathcal{Z}$:

$$\Delta = \partial_x^2 + x^2 \partial_y^2 - \frac{1}{x} \partial_x. \tag{2}$$

It is not hard to show that Δ with domain $\mathrm{Dom}(\Delta) = C_c^\infty(M)$ is essentially self-adjoint on $L^2(M)$, where M is either $\mathbb{R}^2 \setminus \mathcal{Z}$ or one of its two connected components. As a consequence, by Stone Theorem, there exists a unique unitary Schrödinger evolution defined for any initial datum in $L^2(M)$, without the need to impose boundary conditions. From a physical viewpoint this means that quantum particles are naturally confined to stay into M. This is different from what happens, for example, in the case of the Euclidean Laplacian on $\mathbb{R}^2 \setminus \mathcal{Z}$. Indeed, this operator is not essentially self-adjoint and its different self-adjoint extensions correspond to different dynamics, e.g. to complete reflection or transmission of quantum particles at $\mathcal{Z}$, to be chosen depending on the physics of the problem. Similar considerations hold for heat diffusion or wave equations.

The Grushin structure belongs to a class of singular Riemannian structures, called almost-Riemannian structure (ARS), introduced in [3]. The study of essential self-adjointness of the Laplace-Beltrami operator for ARS has been initiated in [6, 7], for surfaces, and in [18], for general dimension. In the latter, as a particular instance of a more general criterion, it has been proved that the metric boundary of a non-complete Riemannian manifold can develop a repulsive effect, quantified in terms of an intrinsic invariant called *effective potential*, whose strength can entail the essential self-adjointness of the Laplace-Beltrami operator [18, Thm. 1].

2010 *Mathematics Subject Classification.* Primary: 47B25, 35J10, 53C21, 58J99; Secondary: 35Q40, 81Q10.

In this paper we extend the results of [18] to a class of natural (Hörmander-type) hypoelliptic operators, the *sub-Laplacians*, arising in sub-Riemannian geometry as a generalization of the Riemannian Laplace-Beltrami operator to this setting.

Roughly speaking, a sub-Riemannian structure on a smooth manifold N is defined by a (possibly rank-varying) smooth distribution $\mathcal{D} \subset TN$ endowed with a scalar product $g : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$. (For a precise definition, see Section 2.) Since the distribution $\mathcal{D}$ is assumed to satisfy the *Lie bracket generating condition*, any two points in N can be joined by curves a.e. tangent to $\mathcal{D}$, of which the scalar product allows to measure the length. As in the Riemannian case, by minimizing the length of such curves one can define a distance d on N. Given a measure ω on N, which is smooth outside of some closed set $\mathcal{Z} \subset N$, the associated sub-Laplacian is the Hörmander-type operator on $L^2(N, \omega)$ defined by

$$\Delta_\omega = \operatorname{div}_\omega \circ \nabla, \qquad \operatorname{Dom}(\Delta_\omega) = C_c^\infty(N \setminus \mathcal{Z}), \tag{3}$$

where the divergence is computed with respect to ω, and ∇ is the sub-Riemannian gradient.

It is well known that if $\mathcal{Z} = \varnothing$ and the sub-Riemannian structure is complete then Δ_ω is essentially self-adjoint on $L^2(N, \omega)$ [23]. Here, we focus on the case of singular measures ω, that is $\mathcal{Z} \neq \varnothing$. In this setting, our main result is the following criterion for essential self-adjointness of sub-Laplacians, that generalizes [18, Thm. 1]. The new aspects of the proof with respect to the results in [18] are the exploitation of subellipticity to obtain regularity properties of weak solutions (Lemma 4.2), and the sub-Riemannian version of the Rellich-Kondrachov theorem (Lemma 4.3).

Theorem 1.1. *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth, embedded, compact hypersurface with no characteristic points. Assume also that, for some $\varepsilon > 0$, there exists a constant $\kappa \geq 0$ such that, letting $\delta = d(\mathcal{Z}, \cdot)$, we have*

$$V_{\text{eff}} = \left(\frac{\Delta_\omega \delta}{2}\right)^2 + \left(\frac{\Delta_\omega \delta}{2}\right)' \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}, \qquad \text{for } 0 < \delta \leq \varepsilon, \tag{4}$$

where the prime denotes the derivative in the direction of $\nabla\delta$. Then Δ_ω with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$, or any of its connected components.

Moreover, if M is relatively compact, the unique self-adjoint extension of Δ_ω has compact resolvent. Therefore, its spectrum is discrete and consists of eigenvalues with finite multiplicity.

Remark 1.1. The compactness of $\mathcal{Z}$ in Theorem 1.1 is not necessary, and it can be replaced by the weaker assumption that the (normal) injectivity radius from $\mathcal{Z}$ is strictly positive.

A particularly interesting case, is the one where the measure ω is chosen to be the *Popp's measure* $\mathcal{P}$. This is a measure canonically associated with the sub-Riemannian structure, which is smooth where the structure is equiregular [16, 5]. In this case, the singular region $\mathcal{Z}$ coincides with the singular region of the sub-Riemannian structure, i.e., the complement of the equiregular region. We refer to the sub-Laplacian $\Delta_\mathcal{P}$ associated with $\mathcal{P}$ as the intrinsic (or Popp) sub-Laplacian.

Consider for example the Martinet structure on $N = \mathbb{R}^3$, whose distribution and metric are defined by the orthonormal vector fields

$$X_1 = \partial_y + x^2 \partial_z, \qquad X_2 = \partial_x. \tag{5}$$

The distribution $\mathcal{D} = \operatorname{span}\{X_1, X_2\}$ is then equiregular everywhere except on the hypersurface $\mathcal{Z} = \{x = 0\}$, where Popp's measure is singular. Indeed,

$$\mathcal{P} = \frac{1}{2\sqrt{2}|x|} dx \wedge dy \wedge dz. \tag{6}$$

In this case, $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(N \setminus \mathcal{Z})$ is essentially self-adjoint. This fact has been proved in [6, Thm. 3] for a compactified version of the Martinet structure on $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$, using a Fourier decomposition w.r.t. the compact singular region $\mathcal{Z} \simeq \mathbb{S}^1 \times \mathbb{S}^1$.

This result has driven the authors to conjecture that the loss of equiregularity acts as a general barrier for quantum diffusion, i.e., more precisely, that the intrinsic sub-Laplacian, when restricted to the equiregular region of a sub-Riemannian manifold, is essentially self-adjoint. As a consequence of Theorem 1.1, we prove this conjecture under mild regularity assumptions on the sub-Riemannian structure (Popp-regularity, see Section 5).

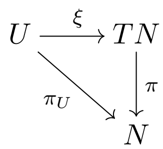
Theorem 1.2. *Let N be a complete and Popp-regular sub-Riemannian manifold, with compact singular set $\mathcal{Z}$. Then, the sub-Laplacian $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$ or one of its connected components. Moreover, if M is relatively compact, the unique self-adjoint extension of $\Delta_{\mathcal{P}}$ has compact resolvent.*

1.1. **Structure of the paper.** The necessary preliminaries of sub-Riemannian geometry are discussed in Section 2. Section 3 is devoted to the proof of regularity properties of the distance function from the singular region. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we discuss the case of the intrinsic sub-Laplacian. We close the paper with examples of non-Popp-regular structures where Theorem 1.2 does not apply, but Theorem 1.1 does, and hence the intrinsic sub-Laplacian is essentially self-adjoint. We also provide examples where both results do not apply, and we are not able to determine whether the sub-Laplacian is essentially self adjoint.

2. Preliminaries on sub-Riemannian geometry

Definition 2.1. Let N be a connected smooth manifold. A *sub-Riemannian structure* on N is a triple $(U, \xi, (\cdot|\cdot)_q)$, where

- $\pi_U : U \to N$ is an Euclidean bundle with base N and Euclidean fiber $U_q = \pi^{-1}(q)$, in particular for every $q \in N$, U_q is a vector space equipped with a scalar product $(\cdot|\cdot)_q$, smooth with respect to q.
- $\xi : U \to TN$ is a vector bundle morphism, i.e., ξ is a fiber-wise linear map such that, letting $\pi : TN \to N$ be the canonical projection, the following diagram commutes:



- The *Lie bracket generating condition* holds true, i.e.,

$$\mathrm{Lie}(\xi(\Gamma(U)))\big|_q = T_qN, \qquad \forall q \in N, \tag{7}$$

where $\Gamma(U)$ denotes the $C^\infty(N)$-module of smooth sections of U and $\mathrm{Lie}(\xi(\Gamma(U)))\big|_q$ denotes the smallest Lie algebra containing $\xi(\Gamma(U)) \subseteq \Gamma(TN)$, evaluated at q.

The subspace of *horizontal directions* at $q \in N$ is $\mathcal{D}_q = \xi(U_q) \subseteq T_qN$ and the set of *horizontal vector fields* is $\Gamma(\mathcal{D}) = \xi(\Gamma(U))$.

Consider a local frame for U, i.e., a set $\sigma_1, \dots, \sigma_m$, with $m = \mathrm{rank}(U)$, of smooth local sections of U, defined on some neighborhood $\mathcal{O} \subseteq N$, and which are orthonormal with respect to the scalar product on U. The vector fields $X_i := \xi \circ \sigma_i$ constitute a *local generating family*. On $\mathcal{O}$, condition (7) reads

$$\mathrm{Lie}(X_1, \dots, X_m)\big|_q = T_qN, \qquad \forall q \in \mathcal{O}. \tag{8}$$

Let $r(q) = \dim(\mathcal{D}_q)$ be the *rank* of the distribution at $q \in N$. Moreover, for $k \in \mathbb{N}$, let

$$\mathcal{D}_q^k := \mathrm{span}\{[X_1, \dots, [X_{j-1}, X_j]]_q : X_i \in \Gamma(\mathcal{D}),\ j \leq k\}. \tag{9}$$

By (7), we call the *step* of the sub-Riemannian structure at q the minimal integer $s = s(q) \in \mathbb{N}$ such that $\mathcal{D}_q^s = T_q N$.

Definition 2.2. Let $A \subseteq N$. We say that a sub-Riemannian structure on N is *equiregular* on A if $\dim(\mathcal{D}_q^k)$ is constant for $q \in A$ and for any $k \in \mathbb{N}$.

Notice that even $r(q) = \dim(\mathcal{D}_q^1)$ can be non-constant. For instance, this is the case of almost-Riemannian manifolds, where there exists a closed set $\mathcal{Z} \subset N$ such that $\dim(\mathcal{D}_q^1) = \dim N$ for every $q \in N \setminus \mathcal{Z}$.

In this paper, N is a smooth manifold without boundary, endowed with a sub-Riemannian structure. Moreover, we let $\mathcal{Z} \subset N$ be a set satisfying

(H0) $\qquad \mathcal{Z} \subseteq N$ is a smooth, embedded hypersurface.

The set $\mathcal{Z}$ will be called the *singular region* when defined in association with a measure ω on N, smooth on $N \setminus \mathcal{Z}$.

Definition 2.3. Let $\mathcal{Z} \subseteq N$ be a smooth embedded hypersurface. We say that $q \in \mathcal{Z}$ is a *characteristic (or tangency) point* if $\mathcal{D}_q \subseteq T_q\mathcal{Z}$.

We will also assume that:

(H1) $\qquad$ The singular region $\mathcal{Z}$ does not contain characteristic points.

Assumption (H1) implies that there are no abnormal minimizers between $p \in N \setminus \mathcal{Z}$ and $\mathcal{Z}$ (see Proposition 2.7). However, we stress that we do not exclude the presence of other abnormal minimizers. (See [1] for a definition of abnormal minimizers.)

2.1. Metric structure.

Let $q \in N$ and $v \in \mathcal{D}_q$. We define the *sub-Riemannian norm* as

$$|v|^2 = \inf\{(u|u)_q : u \in U_q,\ \xi(u) = v\}. \tag{10}$$

One can check that the above norm satisfies the parallelogram law, and hence it is defined by a scalar product on $\mathcal{D}_q$, denoted with the symbol g_q.

Let I be an interval. An *horizontal curve* is an absolutely continuous curve $\gamma : I \to N$ such that $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for a.e. $t \in I$. In this case, we define the *length* of γ as

$$\ell(\gamma) = \int_I |\dot\gamma(t)|\, dt. \tag{11}$$

Since ℓ is invariant by reparametrization of γ, when dealing with minimization of length we consider only intervals of the form $[0, T]$ for some fixed $T > 0$. We define the *sub-Riemannian distance* as

$$d(p, q) := \inf\{\ell(\gamma) : \gamma \text{ is horizontal},\ \gamma(0) = p,\ \gamma(T) = q\}. \tag{12}$$

Under the bracket-generating condition (7), the Chow-Rashevskii Theorem implies that any couple of points $p, q \in N$ can be connected by means of horizontal curves. That is, $d : N \times N \to \mathbb{R}$ is finite. Moreover, d is a continuous map and the metric space (N, d) has the same topology as N.

Definition 2.4. The sub-Riemannian (or horizontal) gradient of a smooth function f is the smooth vector field $\nabla f \in \Gamma(\mathcal{D})$ such that

$$g(\nabla f, W) = df(W), \qquad \forall W \in \Gamma(\mathcal{D}). \tag{13}$$

Remark 2.1. In terms of a local generating family $X_1, \ldots, X_r$ for the sub-Riemannian structure, we have

$$\nabla f = \sum_{i=1}^r X_i(f) X_i, \qquad |\nabla f|^2 = \sum_{i=1}^r X_i(f)^2. \tag{14}$$

Formula (14) holds also if $X_1, \ldots, X_r$ are not independent, in particular it holds on $\mathcal{Z}$.

2.1.1. *Sub-Laplacians.* Let ω be a measure on N, smooth and positive on $N \setminus \mathcal{Z}$. The sub-Laplacian Δ_ω is the operator

$$\Delta_\omega u := \operatorname{div}_\omega(\nabla u), \qquad \forall u \in C_c^\infty(N \setminus \mathcal{Z}), \tag{15}$$

where the divergence $\operatorname{div}_\omega$ is computed with respect to the measure ω, and ∇ is the sub-Riemannian gradient. Equivalently, Δ_ω can be defined as the operator associated with the quadratic form

$$\mathcal{E}(u,v) := \int_M g(\nabla u, \nabla \bar{v})\, d\omega, \qquad \forall v, w \in C_c^\infty(N \setminus \mathcal{Z}). \tag{16}$$

In terms of a local generating family of vector fields $X_1, \dots, X_r \subset \Gamma(\mathcal{D})$, we have

$$\Delta_\omega = \sum_{i=1}^k X_i^2 + \operatorname{div}_\omega(X_i) X_i. \tag{17}$$

As a consequence of the Lie bracket generating assumption, Δ_ω is hypoelliptic [15]. Finally, it is well-known that if $\mathcal{Z} = \varnothing$ and the sub-Riemannian structure is complete then Δ_ω is essentially self-adjoint on $L^2(N)$ [23].

2.1.2. *Geodesics and Hamiltonian flow.* We recall basic notions on minimizing curves in sub-Riemannian geometry. A *geodesic* is a horizontal curve $\gamma : [0,T] \to N$ that locally minimizes the length between its endpoints, and is parametrized by constant speed.

Definition 2.5. The *sub-Riemannian Hamiltonian* is the smooth function $H : T^*N \to \mathbb{R}$,

$$H(\lambda) := \frac{1}{2} \sum_{i=1}^r \langle \lambda, X_i \rangle^2, \qquad \lambda \in T^*N, \tag{18}$$

where $X_1, \dots, X_r$ is a local generating family for the sub-Riemannian structure, and $\langle \lambda, \cdot \rangle$ denotes the action of covectors on vectors. Associated with H we define the *Hamiltonian vector field* $\vec{H}$ on T^*N as $\vec{H} : C^\infty(T^*N) \to C^\infty(T^*N)$ such that $\sigma(\cdot, \vec{H}) = dH$. Here, $\sigma \in \Lambda^2(T^*N)$ is the canonical symplectic form on N.

Solutions $\lambda : [0,T] \to T^*N$ of *Hamilton equations*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)) \tag{19}$$

are called *normal extremals*. Their projections $\gamma(t) := \pi(\lambda(t))$ on N, where $\pi : T^*N \to N$ is the canonical projection, are locally minimizing curves parametrized by constant speed, and are called *normal geodesics*. It is easy to show that if $\lambda(t)$ is a normal extremal, and $\gamma(t) = \pi(\lambda(t))$ is the corresponding normal geodesic, then

$$\dot{\gamma}(t) = \sum_{i=1}^r \langle \lambda(t), X_i(\gamma(t)) \rangle X_i(\gamma(t)), \tag{20}$$

and its speed is given by $|\dot{\gamma}| = \sqrt{2H(\lambda)}$. In particular

$$\ell(\gamma|_{[0,\tau]}) = \sqrt{2H(\lambda(\tau))} \quad \forall \tau \in [0,T]. \tag{21}$$

Definition 2.6. The *exponential map* $\exp_q : D_q \to N$, with base $q \in N$ is

$$\exp_q(\lambda) := \pi \circ e^{\vec{H}}(\lambda), \qquad \lambda \in D_q, \tag{22}$$

where $D_q \subseteq T_q^*N$ is the set of covectors such that the solution $t \mapsto e^{t\vec{H}}(\lambda)$ of (19) with initial datum λ is well defined up to time $T = 1$.

We say that a sub-Riemannian structure on N is complete if (N, d) is a complete metric space. In a complete sub-Riemannian structure, the sub-Riemannian version of Hopf-Rinow theorem implies that $D_q = T_q^*N$ for every $q \in N$.

There is another class of minimizing curves in sub-Riemannian geometry, called *abnormal minimizers*. These curves can be still lifted to extremal curves $\lambda(t)$ on T^*N, but which may not follow the Hamiltonian dynamic of(19). Here we only observe that, in particular, if $\lambda(t) \in T^*N$ is an abnormal extremal, it satisfies:

$$\langle \lambda(t), D_{\pi(\lambda(t))} \rangle = 0 \qquad \forall i = 1, \dots, r, \quad \forall t \in [0, T] \tag{23}$$

with $\lambda(t) \neq 0$ for any $t \in [0, T]$ (see [1, Thm 3.44]), implying $H(\lambda(t)) \equiv 0$. Notice also that a curve may be abnormal and normal at the same time.

Proposition 2.7. *Assume that a sub-Riemannian structure on a smooth manifold N is equiregular outside of a closed embedded hypersurface $\mathcal{Z} \subset N$ Let $\gamma : [0, T] \to N$ be an abnormal minimizer such that $\gamma(0) \in \mathcal{Z}$, $\gamma(T) = p \in N \setminus \mathcal{Z}$ and*

$$\ell(\gamma) = \inf\{d(q, p), \ q \in \mathcal{Z}\}. \tag{24}$$

Then, $\gamma(0) \in \mathcal{Z}$ is a characteristic point.

Proof. By assumption, there exists an abnormal extremal $t \in [0, T] \mapsto \lambda(t)$ such that $\gamma(t) = \pi(\lambda(t))$. The latter satisfies (23). On the other hand, γ minimizes also the distance from $\mathcal{Z}$, hence, by [4, Thm 12.4] the following transversality condition holds true:

$$\langle \lambda(0), v \rangle = 0, \qquad \forall v \in T_{\gamma(0)}\mathcal{Z}. \tag{25}$$

We deduce that $\gamma(0) \in \mathcal{Z}$ is a characteristic point. In fact, if $\mathcal{D}_{\gamma(0)}$ were transversal to $T_{\gamma(0)}\mathcal{Z}$, (25) and (23) would imply that $\lambda(0) = 0$ which is a contradiction. □

2.2. **Popp's measure.** On equiregular neighborhoods of a sub-Riemannian manifold, it is possible to define an intrinsic smooth measure $\mathcal{P}$, called Popp's measure. This measure was introduced first in [16] and then used in [2] to define an intrinsic sub-Laplacian in the sub-Riemannian setting. In the following, we recall the explicit formula for Popp's measure given in [5] in terms of adapted frames, which will be used in Section 5.

Let $\mathcal{O} \subseteq N$ be an equiregular neighborhood of an n-dimensional sub-Riemannian manifold N. A local frame $X_1, \dots, X_n$ on $\mathcal{O}$ is said to be *adapted* to the sub-Riemannian structure if $X_1, \dots, X_{k_i}$ is a local frame for $\mathcal{D}^i$, where $k_i = \dim(\mathcal{D}^i)$ is constant on $\mathcal{O}$. In particular $r(q) \equiv r$ is constant on $\mathcal{O}$. Notice that, the equiregularity assumption means that, on $\mathcal{O}$, $\mathcal{D}^i$ are "true" distributions, and hence that there always exists a local adapted frame. Define the smooth functions $b^\ell_{i_1 \dots i_j} \in C^\infty(N)$ as

$$[X_{i_1}, [X_{i_2}, \dots, [X_{i_{j-1}}, X_j]]] = \sum_{\ell = k_{j-1}+1}^{k_j} b^\ell_{i_1 i_2 \dots i_j} X_\ell \qquad \text{mod } \mathcal{D}^{j-1}, \tag{26}$$

where $1 \leq i_1, \dots, i_j \leq m = \dim(\mathcal{D}^1)$. Consider the $k_j - k_{j-1}$ dimensional square matrices

$$(B_j)^{h\ell} = \sum_{i_1, \dots, i_j = 1}^{r} b^h_{i_1, \dots, i_j} b^\ell_{i_1, \dots, i_j}, \qquad \forall j = 1, \dots, s, \tag{27}$$

where s is the step of the structure. Then, denoting by $\nu^1, \dots, \nu^n$ the dual frame to $X_1, \dots, X_n$, the Popp's measure reads

$$\mathcal{P} = \frac{1}{\sqrt{\prod_{j=1}^s \det B_j}} |\nu^1 \wedge \dots \wedge \nu^n|. \tag{28}$$

One can check that the measure defined by (28) does not depend on the choice of the local adapted frame, and can be taken as the definition of Popp's measure. It is not hard to see, using the very definition, that if $q \in \bar{\mathcal{O}}$ is a non equiregular point, then

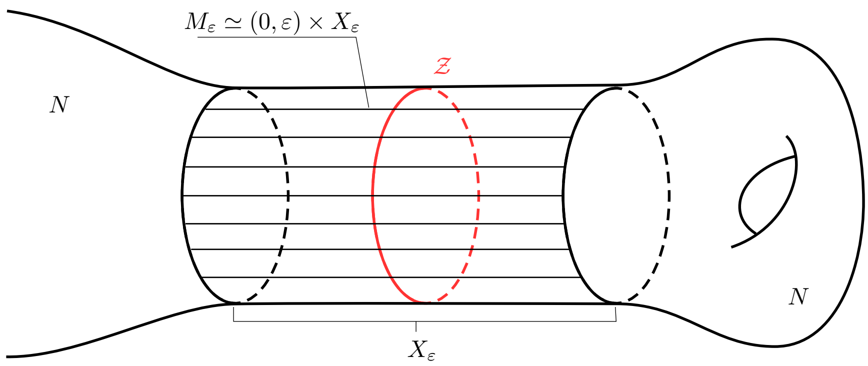


FIGURE 1. Tubular neighborhood of the singular region.

$\lim \sqrt{\prod \det B_j} = 0$ hence the Radon-Nikodym derivative of Popp's measure computed with respect to any globally smooth measure on N diverges to $+\infty$ on the singular region $\mathcal{Z}$. Uniform estimates of this divergence can be found in [14].

3. SUB-RIEMANNIAN DISTANCE FROM AN HYPERSURFACE

We recall that N is a smooth (connected) manifold endowed with a sub-Riemannian structure, and that $\mathcal{Z} \subset N$ is a closed, embedded hypersurface with no characteristic points. We stress that $\mathcal{Z}$ is not necessarily the complement of the equiregular region of the sub-Riemannian structure. The distance from the singular region $\delta : N \to [0,\infty)$ is

$$\delta(p) = \inf\{d(q,p) \mid q \in \mathcal{Z}\}, \qquad \forall p \in N. \tag{29}$$

In the following we resume some basic facts about δ. (See Figure 1.)

Proposition 3.1. *Let N be a smooth sub-Riemannian manifold and $\mathcal{Z} \subset N$ be a smooth, embedded, compact hypersurface with no characteristic points. Then:*

i) *$\delta : N \to [0,\infty)$ is Lipschitz w.r.t. the sub-Riemannian distance and $|\nabla\delta| \leq 1$ a.e.;*
ii) *there exists $\varepsilon > 0$ such that $\delta : M_\varepsilon \to [0,\infty)$ is smooth, where $M_\varepsilon = \{0 < \delta(p) < \varepsilon\}$;*
iii) *letting $X_\varepsilon = \{\delta(p) = \varepsilon\}$, there exists a smooth diffeomorphism $F : (0,\varepsilon) \times X_\varepsilon \to M_\varepsilon$, such that*

$$\delta(F(t,q)) = t \quad \text{and} \quad F_*\partial_t = \nabla\delta, \quad \text{for } (t,q) \in (0,\varepsilon)\times X_\varepsilon. \tag{30}$$

Moreover, $|\nabla\delta| \equiv 1$ on M_ε.

Remark 3.1. The statement and the proof can be simplified if $\mathcal{Z}$ is two-sided (e.g. when N and $\mathcal{Z}$ are orientable). In this case, $M_\varepsilon = (-\varepsilon,0)\times\mathcal{Z} \sqcup (0,\varepsilon)\times\mathcal{Z}$ and there is no need to introduce X_ε. However, this is not true if $\mathcal{Z}$ is one-sided. (For example, think at a Grushin-like structure on the Möbius strip, where $\mathcal{Z}$ is the central line.)

Proof. We prove i). Let $p,q \in N$. By the triangle inequality we have $\delta(p) \leq d(p,q) + \delta(q)$, thus proving that δ is 1-Lipschitz with respect to the sub-Riemannian distance. By [11, Thm. 8] (see also [12, Prop. 2.9], [13, Thm. 1.3]) this implies that the sub-Riemannian gradient satisfies $|\nabla\delta| \leq 1$ almost everywhere.

To prove ii), we follow the same strategy presented in [18, Lemma 7.7]. We first define the *annihilator bundle of the singular set*

$$A\mathcal{Z} := \{(q,\lambda) \in T^*N \mid \lambda(T_q\mathcal{Z}) = 0\}, \tag{31}$$

which is a rank 1 vector bundle with base $\mathcal{Z}$. The map $i_0 : \mathcal{Z} \to A\mathcal{Z}$, $i_0(q) = (q,0)$ is an embedding of $\mathcal{Z}$ onto the zero section of $A\mathcal{Z}$. The bundle $A\mathcal{Z}$ plays the role of the

Riemannian normal bundle usually employed in the construction of a tubular neighborhood. Let $0 \neq \lambda \in A_q\mathcal{Z}$. Since q is not a characteristic point, we have $\lambda(\mathcal{D}_q) \neq 0$. Hence $H(\lambda) > 0$, and the vector

$$v_\lambda = \pi_* \vec{H}(\lambda) = \sum_{i=1}^{r} \langle \lambda, X_i \rangle X_i(q), \tag{32}$$

where $X_1, \dots, X_r$ is a local generating frame of $\mathcal{D}$, is a non-zero horizontal vector transversal to $T_q\mathcal{Z}$. Observe that $\|v_\lambda\|^2 = \langle \lambda, v_\lambda \rangle = 2H(\lambda) > 0$, even if $X_1, \dots, X_r$ are not independent at q.

Let $D \subseteq T^*N$ be the set of (q, λ) such that $\exp_q(\lambda)$ is well defined. Indeed, D is open and so is $D \cap A\mathcal{Z}$ as a subset of $A\mathcal{Z}$. Consider the map $E : A\mathcal{Z} \cap D \to N$, given by

$$E(q, \lambda) := \exp_q(\lambda) = \pi \circ e^{\vec{H}}(\lambda). \tag{33}$$

Claim 1. Given $q \in \mathcal{Z}$, E is a diffeomorphism on a neighborhood $U(q) \subseteq D \cap A\mathcal{Z}$ of $i_0(q) = (q, 0) \in A\mathcal{Z}$.
To prove Claim 1, we first notice that $i_0(\mathcal{Z}) \subseteq D$, and $E \circ i_0 = \mathrm{id}_{\mathcal{Z}}$. Moreover, E has full rank on $i_0(\mathcal{Z})$. In fact, identifying $T_{(q,0)}A\mathcal{Z} \simeq T_q\mathcal{Z} \oplus A_q\mathcal{Z}$, we have $d_{(q,0)}E|_{T_q\mathcal{Z}} = \mathrm{id}_{T_q\mathcal{Z}}$ and for $\delta\lambda \in A_q\mathcal{Z}$

$$d_{(q,0)}E(\delta\lambda) = \sum_{i=1}^{r} \langle \delta\lambda, X_i \rangle X_i = v_{\delta\lambda} \neq 0. \tag{34}$$

Claim 1 now follows from the inverse function theorem and from the fact that $\dim(A\mathcal{Z}) = \dim(N)$. Moreover, since $\mathcal{Z}$ is embedded, and $2H$, restricted to the fibers of $A\mathcal{Z}$, is a well defined norm, the neighborhood $U(q)$ can be taken of the form

$$U(q) = U_\varrho(q) = \{(q', \lambda') \mid d(q, q') < \varrho,\ \sqrt{2H(\lambda')} < \varrho\}, \qquad \varrho > 0. \tag{35}$$

For any $q \in \mathcal{Z}$, let

$$\varepsilon(q) := \sup\{\varrho > 0 \mid E : U_\varrho(q) \to E(U_\varrho(q)) \text{ is a diffeomorphism}\} > 0. \tag{36}$$

Claim 2. The function $\varepsilon : \mathcal{Z} \to \mathbb{R}_+$ is continuous, that is,

$$|\varepsilon(q) - \varepsilon(q')| \leq d(q, q'), \qquad \forall q, q' \in \mathcal{Z}. \tag{37}$$

To prove it, assume without loss of generality that $\varepsilon(q) \geq \varepsilon(q')$. If $d(q, q') \geq \varepsilon(q)$, then (37) holds. On the other hand, if $d(q, q') < \varepsilon(q)$, the triangle inequality for d implies that that $U_\varrho(q') \subseteq U_{\varepsilon(q)}(q)$ for $\varrho = \varepsilon(q) - d(q, q')$, implying Claim 2.

Thanks to the compactness[1] of $\mathcal{Z}$, we define the open neighborhood of $i_0(\mathcal{Z})$:

$$U := \{(q, \lambda) \in A\mathcal{Z} \mid \sqrt{2H(\lambda)} < \varepsilon_0\}, \qquad \varepsilon_0 := \min\{\varepsilon(q)/2 \mid q \in \mathcal{Z}\} > 0. \tag{38}$$

Claim 3. The restriction of E to U is injective.
This follows from the fact that for $(q_1, \lambda_1), (q_2, \lambda_2) \in U$, if $\varepsilon(q_1) \leq \varepsilon(q_2)$, then $(q_1, \lambda_1) \in U_{\varepsilon(q_2)}(q_2)$ (on which E is a diffeomorphism by Claim 1).

By Claim 3, $E : U \to E(U)$ is a smooth diffeomorphism and $E(U) \subseteq \{\delta < \varepsilon_0\}$. Up to taking a smaller ε_0, we can assume that $E(U) \subseteq \{\delta < \varepsilon_0\} \subseteq K$, where K is compact.

Claim 4. $E(U) = \{\delta < \varepsilon_0\}$ and, on $E(U)$, the sub-Riemannian distance from $\mathcal{Z}$ satisfies

$$\delta(E(q, \lambda)) = \sqrt{2H(\lambda)}. \tag{39}$$

To prove Claim 4, let $p \in \{\delta < \varepsilon_0\} \subseteq K$. Since K is compact, there exists at least one horizontal curve $\gamma : [0, 1] \to N$ minimizing the sub-Riemannian distance between $\mathcal{Z}$ and p.

[1] In view of Remark 4.1, we notice that the function $\varepsilon(q)$ is the sub-Riemannian version of the normal injectivity radius from $\mathcal{Z}$ at q, and thus $\inf_q \varepsilon(q)$ is the normal injectivity radius from $\mathcal{Z}$. Hence, if $\mathcal{Z}$ is not compact, we can still proceed by assuming that the normal injectivity radius from $\mathcal{Z}$ is strictly positive.

By Proposition 2.7, this must be a normal geodesic, that is $p = E(q,\lambda)$, with $q \in \mathcal{Z}$ and $\lambda \in T_q^*N$. Since γ is minimizing, transversality conditions (25) imply that $\lambda(T_q\mathcal{Z}) = 0$, that is $(q,\lambda) \in A\mathcal{Z}$. Moreover, $\sqrt{2H(\lambda)} = \ell(\gamma) = \delta(p) < \varepsilon_0$. This implies that $(q,\lambda) \in U$, that is $p = E(q,\lambda) \in E(U)$, and $\delta(E(q,\lambda)) = \sqrt{2H(\lambda)}$, as claimed. Since $\sqrt{2H(\lambda)}$ is a smooth function for $H(\lambda) \neq 0$, δ is smooth on $\{0 < \delta < \varepsilon\}$, for all $\varepsilon \leq \varepsilon_0$.

We prove statement iii). Let $0 < \varepsilon < \varepsilon_0$ and let $F : (0,\varepsilon) \times X_\varepsilon \to M_\varepsilon$ be defined by

$$F(t,q) = E\left(q_0, \frac{t}{\sqrt{2H(\lambda)}}\lambda\right) \tag{40}$$

where, for $q \in X_\varepsilon$, we are using Claim 4 to write $q = E(q_0,\lambda)$ for a unique $(q_0,\lambda) \in U$ such that $\sqrt{2H(\lambda)} = \varepsilon$. The function F is a smooth diffeomorphism, with inverse

$$F^{-1}(p) = \left(\sqrt{2H(\nu)}, E\Big(p_0, \frac{\varepsilon}{\sqrt{2H(\nu)}}\nu\Big)\right), \quad \text{for } p = E(p_0,\nu) \in M_\varepsilon,\ (p_0,\nu) \in U. \tag{41}$$

Moreover, by (39) and the definition of F

$$\delta(F(t,q)) = \sqrt{2H\left(\frac{t}{\sqrt{2H(\lambda)}}\lambda\right)} = t, \qquad \forall (t,q) \in (0,\varepsilon) \times X_\varepsilon. \tag{42}$$

Notice that F is the gradient flow of δ on M_ε. Now, for $q \in X_\varepsilon$, the curves $t \mapsto F(t,q)$ are the unique normal geodesics with speed 1 that minimize the sub-Riemannian distance from $\mathcal{Z}$. Hence, $F_*\partial_t$ is a horizontal vector field and $|F_*\partial_t| = 1$. We conclude the proof by showing that that $\nabla\delta = F_*\partial_t$. In fact, by Cauchy-Schwarz inequality, if $\nabla\delta$ is not parallel to $F_*\partial_t$, then $1 = |g(F_*\partial_t, \nabla\delta)| < |\nabla\delta|$ at some point $F(\bar{t},\bar{q})$. On the other hand, the unit-speed curve $\gamma(s) = e^{s\nabla\delta/|\nabla\delta|}F(\bar{t},\bar{q})$ satisfies, for T small enough,

$$\begin{aligned}\delta(\gamma(T)) - \delta(\gamma(0)) &= \int_0^T \frac{d}{ds}\delta(\gamma(s))\Big|_{s=t} dt = \int_0^T g(\nabla\delta(\gamma(t)), \dot{\gamma}(t))dt \\ &= \int_0^T g\left(\nabla\delta, \frac{\nabla\delta}{|\nabla\delta|}\right)dt = \int_0^T |\nabla\delta| > T = \ell(\gamma|_{[0,T]}),\end{aligned} \tag{43}$$

leading to a contradiction, and implying the statement. □

4. Main quantum completeness criterion

Let N be a complete sub-Riemannian manifold and $\mathcal{Z} \subset N$ be a smooth embedded hypersurface with no characteristic points. Let ω be a measure on N, smooth on $M = N \setminus \mathcal{Z}$ or one of its connected components. We are interested in the essential self-adjointness of the operator

$$H = -\Delta_\omega = -\operatorname{div}_\omega \cdot \nabla, \qquad \operatorname{Dom}(H) = C_c^\infty(M). \tag{44}$$

In the following, we denote with $L^2(M)$ the complex Hilbert space of (equivalence classes of) functions $u : M \to \mathbb{C}$, with scalar product

$$\langle u, v\rangle = \int_M u\bar{v}\, d\omega, \qquad u, v \in L^2(M), \tag{45}$$

where the bar denotes complex conjugation. The corresponding norm is $\|u\|^2 = \langle u, u\rangle$. Similarly, given a coordinate neighborhood $U \subseteq M$ and denoting by dx the Lebesgue measure on it, we denote by $L^2(U, dx)$ the complex Hilbert space of square-integrable functions $u : U \to \mathbb{C}$ satisfying (45) with $d\omega$ replaced by dx and M by U.

Our main result is the following.

Theorem 4.1 (Main quantum completeness criterion)**.** *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth, embedded, compact hypersurface with no characteristic points. Assume also that, for some $\varepsilon > 0$, there exists a constant $\kappa \geq 0$ such that, letting $\delta = d(\mathcal{Z}, \cdot)$, we have*

$$V_{\text{eff}} = \left(\frac{\Delta_\omega \delta}{2}\right)^2 + \left(\frac{\Delta_\omega \delta}{2}\right)' \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}, \qquad \text{for } 0 < \delta \leq \varepsilon, \tag{46}$$

where the prime denotes the derivative in the direction of $\nabla \delta$. Then Δ_ω with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$, or any of its connected components.

Moreover, if M is relatively compact, the unique self-adjoint extension of Δ_ω has compact resolvent. Therefore, its spectrum is discrete and consists of eigenvalues with finite multiplicity.

Remark 4.1. The compactness of $\mathcal{Z}$ in Theorem 4.1 can be replaced by the weaker assumption that the (normal) injectivity radius from $\mathcal{Z}$ is strictly positive. Indeed, in this case, Proposition 3.1 and the forthcoming Proposition 4.6 still hold true. (See footnote in the proof of Proposition 3.1.)

We start by showing two functional results holding on any sub-Riemannian manifold M equipped with a smooth measure ω.

We denote by $W^1(M)$ the Sobolev space of functions in $L^2(M)$ with distributional (sub-Riemannian) gradient $\nabla u \in L^2(\mathcal{D})$, where the latter is the complex Hilbert space of sections of the complexified distribution $X : M \to \mathcal{D}^{\mathbb{C}} \subseteq TM^{\mathbb{C}}$, with scalar product

$$\langle X, Y \rangle = \int_M g(X, Y)\, d\omega, \qquad X, Y \in L^2(\mathcal{D}). \tag{47}$$

The Sobolev space $W^1(M)$ is a Hilbert space when endowed with the scalar product

$$\langle u, v \rangle_{W^1} = \langle \nabla u, \nabla v \rangle + \langle u, v \rangle. \tag{48}$$

Similarly, given a coordinate neighborhood $U \subseteq M$ and denoting by dx the Lebesgue measure on it, we denote by $W^1(U, dx)$ the Sobolev space of functions in $L^2(U, dx)$, with distributional (sub-Riemannian) gradient in $L^2(\mathcal{D}|_U, dx)$, that is the complex Hilbert space of sections of the of the complexified distribution $X : U \to \mathcal{D}^{\mathbb{C}} \subseteq TM^{\mathbb{C}}$, with the scalar product defined in (47) where $d\omega$ is replaced by dx. Moreover, we denote by $L^2_{\text{loc}}(M)$ and $W^1_{\text{loc}}(M)$ the space of functions $u : M \to \mathbb{C}$ such that, for any relatively compact domain $\Omega \subseteq M$, their restriction to Ω belongs to $L^2(\Omega)$ and $W^1(\Omega)$, respectively.

Lemma 4.2. *Let M be a sub-Riemannian manifold equipped with a smooth measure ω. Then* $\mathrm{Dom}(H^*) \subseteq W^1_{\text{loc}}(M)$*.*

Proof. If $u \in \mathrm{Dom}(H^*)$, then $\Delta_\omega u \in L^2_{\text{loc}}(M)$. Let $f = \Delta_\omega u$ and let U be a relatively compact coordinate domain of M. Then, $f \in L^2(U)$, and, since ω is a smooth measure on $N \setminus \mathcal{Z}$, $f \in L^2(U, dx)$, where dx denotes the Lebesgue measure on U. Notice that Δ_ω can be written in the form $\mathcal{L} = \sum_{i=1}^r X_i^2 + X_0$ where $X_1, \dots, X_r$ is a local generating family and X_0 is a horizontal vector field. Then, by Rotschild and Stein subellipticity theory for $\mathcal{L}$ (see [22, Thm. 18.d]), $u \in W^1_{\text{loc}}(U, dx)$, implying $u \in W^1_{\text{loc}}(U)$. We deduce that $u \in W^1_{\text{loc}}(M)$. In fact, if $K \subseteq M$ is a relatively compact domain, we can cover it with a finite number of coordinate charts $U_1, \dots, U_m$, with $K \cap U_i$ relatively compact. In particular, $u \in W^1(K \cap U_i)$ for any $i = 1, \dots, m$, implying $u \in W^1_{\text{loc}}(M)$. □

Lemma 4.3 (Sub-Riemannian Rellich-Kondrachov theorem)**.** *Let M be a sub-Riemannian manifold equipped with a smooth measure ω. Let $\Omega \subseteq M$ be a relatively compact domain with Lipschitz boundary. Then $W^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.*

Proof. Step 1. Let $U \subseteq N$ be a coordinate neighborhood such that $U \cap \Omega$ has Lipschitz boundary and let w_j be a sequence bounded in $W^1(\Omega)$. Since ω is smooth outside $\mathcal{Z}$, this is equivalent to say that w_j is bounded in $W^1(U \cap \Omega, dx)$, where dx denotes the Lebesgue measure on U. By [22, Thm 13] (and estimates therein), if s denotes the step of the sub-Riemannian structure ouside $\mathcal{Z}$, $W^1(U \cap \Omega, dx)$ is compactly embedded into the isotropic fractional Sobolev space $W^{1/s,2}_{\text{iso}}(U \cap \Omega, dx)$. This is defined considering fractional derivatives in every coordinate direction (and not just in the horizontal ones). Then, by the classical Rellich-Kondrachov theorem applied to set $U \cap \Omega$, whose boundary is Lipschitz, we can extract a subsequence w_{j_ℓ} of w_j converging in $L^2(U \cap \Omega, dx)$, hence in $L^2(\Omega \cap U)$.

Step 2. Let u_j be a sequence bounded in $W^1(\Omega)$ and let $\Omega = \bigcup_{\ell=1}^N U_\ell$ be a covering of Ω where each U_ℓ is a coordinate domain. Then u_j is bounded in $W^1(U_\ell)$ for every ℓ. Without loss of generality we can assume $U_\ell \cap \Omega$ to have Lipschitz boundary for every ℓ. By Step 1, we can extract from u_j a subsequence $u_{j_1(k)}$ converging in $L^2(\Omega \cap U_1)$. Similarly, from $u_{j_1(k)}$ we extract a subsequence $u_{j_2(k)}$ converging in $L^2(\Omega \cap U_2)$. By repeating this procedure for every ℓ we obtain a subsequence $u_{j_N(k)}$ of u_j converging in $L^2(\Omega \cap U_\ell)$ for every $\ell = 1, \dots, N$. This implies that $u_{j_N(k)}$ converges in $L^2(\Omega)$, as claimed. □

4.1. **Agmon-type estimates and weak Hardy inequality.** Recall that the symmetric bilinear form associated with H is

$$\mathcal{E}(u, v) = \int_M g(\nabla u, \nabla v)\, d\omega, \qquad u, v \in C^\infty_c(M). \tag{49}$$

We use the same symbol to denote the above integral, eventually equal to $+\infty$, for all functions $u, v \in W^1_{\text{loc}}(M)$. We also let, for brevity, $\mathcal{E}(u) = \mathcal{E}(u, u)$.

Lemma 4.4. *Let M be a sub-Riemannian manifold equipped with a smooth measure ω. Let f be a real-valued function, Lipschitz w.r.t. the sub-Riemannian distance. Let $u \in W^1_{\text{loc}}(M)$, and assume that f or u have compact support $K \subset M$. Then, we have*

$$\mathcal{E}(fu, fu) = \operatorname{Re} \mathcal{E}(u, f^2 u) + \langle u, |\nabla f|^2 u\rangle. \tag{50}$$

Moreover, if $\psi \in \operatorname{Dom}(H^*)$ satisfies $H^*\psi = E\psi$, and f has compact support, we have

$$\mathcal{E}(f\psi, f\psi) = E\|f\psi\|^2 + \langle \psi, |\nabla f|^2 \psi\rangle. \tag{51}$$

Proof. Observe that $|\nabla f|$ is essentially bounded by [11, Thm. 8] (see also [12, Prop. 2.9], [13, Thm. 1.3]). Hence $fu \in W^1_{\text{comp}}(M)$. By using the fact that f is real-valued, a straightforward application of Leibniz rule yields

$$\langle \nabla u, \nabla(f^2 u)\rangle = \langle f\nabla u, \nabla(fu)\rangle + \langle \nabla u, fu\nabla f\rangle \tag{52}$$
$$= \langle \nabla(fu), \nabla(fu)\rangle - \langle u\nabla f, \nabla(fu)\rangle + \langle \nabla u, fu\nabla f\rangle \tag{53}$$
$$= \langle \nabla(fu), \nabla(fu)\rangle - \langle u\nabla f, u\nabla f\rangle - \langle u\nabla f, f\nabla u\rangle + \langle f\nabla u, u\nabla f\rangle \tag{54}$$
$$= \langle \nabla(fu), \nabla(fu)\rangle - \langle u, |\nabla f|^2 u\rangle + 2i \operatorname{Im}\langle f\nabla u, u\nabla f\rangle. \tag{55}$$

Thus, by definition of $\mathcal{E}$, we have

$$\operatorname{Re} \mathcal{E}(u, f^2 u) = \langle \nabla(fu), \nabla(fu)\rangle - \langle u, |\nabla f|^2 u\rangle = \mathcal{E}(fu, fu) - \langle u, |\nabla f|^2 u\rangle, \tag{56}$$

completing the proof of (50).

To prove (51), recall that, by Lemma 4.2, $\operatorname{Dom}(H^*) \subseteq W^1_{\text{loc}}(M)$. Then we obtain

$$\mathcal{E}(u, f^2 u) = \langle \nabla u, \nabla(f^2 u)\rangle = \langle -\Delta_\omega u, f^2 u\rangle = \langle H^* u, f^2 u\rangle. \tag{57}$$

Setting $u = \psi$, we obtain $\mathcal{E}(\psi, f^2\psi) = E\|f\psi\|^2$, yielding the statement. □

We show how to compute V_{eff} through the diffeomorphism F given by Proposition 3.1.

Proposition 4.5. *Using the diffeomorphism of Proposition 3.1 to identify $M_\varepsilon \simeq (0,\varepsilon) \times X_\varepsilon$, we have*

$$d\omega(t,q) = e^{2\theta(t,q)} dt\, d\mu(q), \qquad (t,q) \in M_\varepsilon, \tag{58}$$

where $d\mu$ is a fixed smooth measure on X_ε, and θ is a smooth function. Moreover,

$$V_{\mathrm{eff}} = (\partial_t \theta)^2 + \partial_t^2 \theta. \tag{59}$$

Proof. We prove (59). Through the identification $M_\varepsilon \simeq (0,\varepsilon) \times X_\varepsilon$ we have $\nabla\delta(t,q) = \partial_t$. Then, by definition of $\operatorname{div}_\omega$ we have

$$\begin{aligned} \Delta_\omega \delta(t,q))\omega &= \operatorname{div}_\omega(\partial_t)\omega = \mathcal{L}_{\partial_t}\omega = \mathcal{L}_{\partial_t}(e^{2\theta(t,q)} dt\, d\mu(q)) \\ &= 2\partial_t\theta(t,q) d\omega + e^{2\theta}\mathcal{L}_{\partial_t}(dt\, d\mu(q)) = 2\partial_t\theta(t,q) d\omega, \end{aligned} \tag{60}$$

where we used $\mathcal{L}_{\partial_t}(dt\ d\mu(q)) = 0$. Moreover, in these coordinates, derivation in the direction of $\nabla\delta$ amounts to the derivation w.r.t. t, hence

$$(\Delta_\omega \delta(t,q))' = 2\partial_t^2\theta. \tag{61}$$

□

Proposition 4.6 (Weak Hardy Inequality)**.** *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $M = N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth, embedded, compact hypersurface with no characteristic points. Assume also that there exist $\kappa \geq 0$ and $\varepsilon > 0$ such that,*

$$V_{\mathrm{eff}} \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}, \qquad \text{for } \delta \leq \varepsilon. \tag{62}$$

Then, there exist $\eta \leq 1/\kappa$ and $c \in \mathbb{R}$ such that

$$\int_M |\nabla u|^2\, d\omega \geq \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right) |u|^2\, d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\mathrm{comp}}(M), \tag{63}$$

where $M_\eta = \{0 < \delta < \eta\}$. In particular, the operator $H = -\Delta_\omega$ is semibounded on $C_c^\infty(M)$.

Proof. By Proposition 3.1 there exists $\varepsilon > 0$ such that δ is smooth on $M_\varepsilon = \{0 < \delta < \varepsilon\}$.

First we prove (63) for $u \in W^1_{\mathrm{comp}}(M_\varepsilon)$, and with $\eta = \varepsilon$, possibly not satisfying $\eta \leq 1/\kappa$. Then, we extend it for $u \in W^1_{\mathrm{comp}}(M)$, choosing $\eta \leq 1/\kappa$.

Step 1. Let $u \in W^1_{\mathrm{comp}}(M_\varepsilon)$. By Proposition 3.1, we identify $M_\varepsilon \simeq (0,\varepsilon) \times \mathcal{Z}$ in such a way that $\delta(t,q) = t$. By Proposition 4.5, fixing a reference measure $d\mu$ on $\mathcal{Z}$, we have $d\omega(t,q) = e^{2\vartheta(t,q)} dt\, d\mu(q)$ on M_ε, for some smooth function $\vartheta : M_\varepsilon \to \mathbb{R}$. Consider the unitary transformation $T : L^2(M_\varepsilon, d\omega) \to L^2(M_\varepsilon, dt\, d\mu)$ defined by $Tu = e^{\vartheta} u$. By Proposition 3.1 ∂_t is a unit horizontal vector field. Hence $|\nabla u| \geq |\partial_t u|$. Letting $v = Tu$, an integration by parts yields

$$\int_M |\nabla u|^2\, d\omega \geq \int_{M_\varepsilon} |\partial_t u|^2\, d\omega = \int_{M_\varepsilon} \left(|\partial_t v|^2 + \Big(\underbrace{(\partial_t\vartheta)^2 + \partial_t^2\vartheta}_{=V_{\mathrm{eff}}} \Big) |v|^2 \right) dt\, d\mu, \tag{64}$$

where the expression for V_{eff} is in Proposition 4.5. Recall the 1D Hardy inequality:

$$\int_0^\varepsilon |f'(s)|^2\, ds \geq \frac{1}{4}\int_0^\varepsilon \frac{|f(s)|^2}{s^2}\, ds, \qquad \forall f \in W^1_{\mathrm{comp}}((0,\varepsilon)). \tag{65}$$

Since $u \in W^1_{\mathrm{comp}}(M_\varepsilon)$ and ϑ is smooth, for a.e. $q \in X_\varepsilon$, the function $t \mapsto v(t,q)$ is in $W^1_{\mathrm{comp}}((0,\varepsilon))$ (see [10, Thm. 4.21]). Then, by using (62), Fubini's Theorem and (65), we obtain (63) for functions $u \in W^1_{\mathrm{comp}}(M_\varepsilon)$ with $\eta = \varepsilon$ and $c = 0$.

Step 2. Let $u \in W^1_{\mathrm{comp}}(M)$, and let χ_1, χ_2 be smooth functions on $[0,+\infty)$ such that

- $0 \leq \chi_i \leq 1$ for $i = 1,2$;
- $\chi_1 \equiv 1$ on $[0, \frac{\varepsilon}{2}]$ and $\chi_1 \equiv 0$ on $[\varepsilon, +\infty)$;
- $\chi_2 \equiv 0$ on $[0, \frac{\varepsilon}{2}]$ and $\chi_2 \equiv 1$ on $[\varepsilon, +\infty)$;

- $\chi_1^2 + \chi_2^2 = 1$.

Consider the functions $\phi_i : M \to \mathbb{R}$ defined by $\phi_i := \chi_i \circ \delta$. We have $\phi_1 \equiv 1$ on $M_{\varepsilon/2}$, $M_{\varepsilon/2} \subseteq \mathrm{supp}(\phi_1) \subseteq M_\varepsilon$, moreover $0 \le \phi_1 \le 1$, and $\phi_1^2 + \phi_2^2 = 1$. Notice that $\phi_2 \equiv 1$ and $\phi_1 \equiv 0$ on $M \setminus M_\varepsilon$, and so $\nabla \phi_i \equiv 0$ there. Moreover, since by Proposition 3.1 i) there holds $|\nabla \delta| \le 1$ a.e., we have

$$c_1 = \sup_M \sum_{i=1}^2 |\nabla \phi_i|^2 \le \sup_{[0,\varepsilon]} \sum_{i=1}^2 |\chi_i'|^2 < +\infty. \tag{66}$$

By (50) of Lemma 4.4, we obtain

$$\mathcal{E}(u) = \sum_{i=1}^2 \mathcal{E}(\phi_i u) - \sum_{i=1}^2 \int_M |\nabla \phi_i|^2 |u|^2 d\omega \ge \mathcal{E}(\phi_1 u) - c_1 \|u\|^2, \tag{67}$$

where we used that $\mathcal{E}(\phi_2 u) \ge 0$, that $\phi_1^2 + \phi_1^2 = 1$ and the inequality (66). In particular, applying the statement proved in Step 1 to $\phi_1 u \in W^1_{\mathrm{comp}}(M_\varepsilon)$, we get

$$\mathcal{E}(u) \ge \int_{M_\varepsilon} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |\phi_1 u|^2 d\omega - c_1 \|u\|^2. \tag{68}$$

Letting $\eta = \min\{\frac{\varepsilon}{2}, 1/\kappa\}$, we have

$$\mathcal{E}(u) \ge \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 d\omega - \int_{M_\varepsilon \setminus M_\eta} \left| \frac{1}{\delta^2} - \frac{\kappa}{\delta} \right| |\phi_1 u|^2 d\omega - c_1 \|u\|^2 \tag{69}$$

$$\ge \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 d\omega - \left(c_1 + \sup_{\eta \le \delta \le \varepsilon} \left| \frac{1}{\delta^2} - \frac{\kappa}{\delta} \right| \right) \|u\|^2, \tag{70}$$

which concludes the proof. □

Proposition 4.7 (Agmon-type estimate). *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $M = N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth embedded hypersurface with no characteristic points. Assume also that there exist $\kappa \ge 0$, $\eta \le 1/\kappa$ and $c \in \mathbb{R}$ such that,*

$$\int_M |\nabla u|^2 \, d\omega \ge \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\mathrm{comp}}(M). \tag{71}$$

Then, for all $E < c$, the only solution of $H^*\psi = E\psi$ is $\psi \equiv 0$.

Notice that the requirement $\eta \le 1/\kappa$ ensures the non-negativity of the integrand in (71). The proof follows the ideas of [17, 9].

Proof. Let $f : M \to \mathbb{R}$ be a bounded Lipschitz function w.r.t. the sub-Riemannian distance with $\mathrm{supp}\, f \subseteq \overline{M \setminus M_\zeta}$, for some $\zeta > 0$, and ψ be a solution of $(H^* - E)\psi = 0$ for some $E < c$. We start by claiming that

$$(c - E)\|f\psi\|^2 \le \langle \psi, |\nabla f|^2 \psi \rangle - \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |f\psi|^2 d\omega. \tag{72}$$

If f had compact support, then $f\psi \in W^1_{\mathrm{comp}}(M)$, and hence (72) would follow directly from (71) and (51). To prove the general case, let $\theta : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\theta(s) = \begin{cases} 1 & s \le 0, \\ 1 - s & 0 \le s \le 1, \\ 0 & s \ge 1. \end{cases} \tag{73}$$

Fix $q \in M$ and let $G_n : M \to \mathbb{R}$ defined by $G_n(p) = \theta(d_g(q,p) - n)$. Notice that G_n is Lipschitz w.r.t. the sub-Riemannian distance, and hence its sub-Riemannian gradient satisfies $|\nabla G_n| \leq 1$, see [11, Thm. 8], [12, Prop. 2.9], [13, Thm. 1.3]. Moreover $\mathrm{supp}(G_n) \subseteq \bar{B}_q(n+1)$. Observe that

$$\mathrm{supp}\, G_n f \subseteq \overline{(M \setminus M_\zeta) \cap B_q(n+1)}. \tag{74}$$

Even if (M,d) is a non-complete metric space (and hence, its closed balls might fail to be compact), the set on the right hand side of (74) is compact, being uniformly separated from the metric boundary. This can be proved with the same argument of [8, Prop. 2.5.22] and exploiting the completeness of (N,d). Hence, the support of $f_n := G_n f$ is compact, and (72) holds with f_n in place of f. The claim now follows by dominated convergence. Indeed, $f_n \to f$ point-wise as $n \to +\infty$ and $f_n \leq f$. Hence $\|f_n\psi\| \to \|f\psi\|$. Thus, since $\mathrm{supp}\, f_n \subseteq \overline{M \setminus M_\zeta}$, we have

$$\lim_{n\to+\infty} \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right) |f_n\psi|^2 \, d\omega = \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right) |f\psi|^2 \, d\omega. \tag{75}$$

Finally, since $|\nabla f_n| \leq C$, and $\nabla f_n \to \nabla f$ a.e. we have $\langle \psi, |\nabla f_n|^2 \psi\rangle \to \langle \psi, |\nabla f|^2\psi\rangle$, yielding the claim.

We now plug a particular choice of f into (72). Set

$$f(p) := \begin{cases} F(\delta(p)) & 0 < \delta(p) \leq \eta, \\ 1 & \delta(p) > \eta, \end{cases} \tag{76}$$

where F is a Lipschitz function to be chosen later. Recall that $|\nabla\delta| \leq 1$ a.e. on M. In particular, a.e. on M_η, we have $|\nabla f| = |F'(\delta)||\nabla\delta| \leq |F'(\delta)|$. Thus, by (72), we have

$$(c - E)\|f\psi\|^2 \leq \int_{M_\eta} \left[F'(d)^2 - \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right) F(d)^2\right] |\psi|^2 d\omega. \tag{77}$$

Let now $0 < 2\zeta < \eta$. We choose F for $\tau \in [2\zeta, \eta]$ to be the solution of

$$F'(\tau) = \sqrt{\frac{1}{\tau^2} - \frac{\kappa}{\tau}} F(\tau), \qquad \text{with } F(\eta) = 1, \tag{78}$$

to be zero on $[0,\zeta]$, and linear on $[\zeta, 2\zeta]$. Observe that the assumption $\eta \leq 1/\kappa$ implies that (78) is well defined. We first consider the case $\kappa = 0$. The function F, together with its derivative reads

$$F(t) = \begin{cases} 0 & t \in [0,\zeta], \\ \frac{2}{\eta}(t-\zeta) & t \in [\zeta, 2\zeta], \\ \frac{1}{\eta} t & t \in [2\zeta, \eta), \end{cases} \qquad F'(t) = \begin{cases} 0 & t \in [0,\zeta], \\ \frac{2}{\eta} & t \in [\zeta, 2\zeta], \\ \frac{1}{\eta} & t \in [2\zeta, \eta). \end{cases} \tag{79}$$

The global function defined by (76) is a Lipschitz function with support contained in $\overline{M \setminus M_\zeta}$ and such that $F' \leq K$ on $[\zeta, 2\zeta]$, for some constant independent of ζ ($K = 2/\eta$). Therefore, from (77) we get

$$(c-E)\|f\psi\|^2 \leq \int_{M_{2\zeta}\setminus M_\zeta} \left[F'(d)^2 - \frac{1}{\delta^2} F(d)^2\right] |\psi|^2 d\omega \leq K^2 \int_{M_{2\zeta}\setminus M_\zeta} |\psi|^2 d\omega. \tag{80}$$

If we let $\zeta \to 0$, then f tends to an almost everywhere strictly positive function. Recalling that $E < c$, and taking the limit, equation (80) implies $\psi \equiv 0$. When $\kappa > 0$ the solution to (78), on the interval $[2\zeta, \eta]$, is

$$F(\tau) = C(\kappa,\eta) \frac{1 - \sqrt{1-\kappa\tau}}{1 + \sqrt{1-\kappa\tau}} e^{2\sqrt{1-\kappa\tau}}, \qquad \tau \in [2\zeta, \eta], \tag{81}$$

for a constant $C(\kappa,\eta)$ such that $F(\eta) = 1$. By construction of F on $[\zeta, 2\zeta]$, we obtain

$$F'(\tau) = \frac{F(2\zeta)}{\zeta}, \qquad \tau \in [\zeta, 2\zeta]. \tag{82}$$

Hence we have $F(2\zeta) = C(\kappa,\eta)e^2\kappa\zeta/2 + o(\zeta)$, which yields the boundedness of F' on $[\zeta, 2\zeta]$ by a constant not depending on ζ. Moreover the global function defined by (76) is Lipschitz with support contained in $\overline{M\setminus M_\zeta}$. Thus, by (77), we conclude that $\|\psi\| = 0$ as in the case $\kappa = 0$. ☐

Remark 4.2 (The role of the Hardy constant in the proof). If (71) is replaced with

$$\int_M |\nabla u|^2\, d\omega \geq a \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right) |u|^2 d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\mathrm{comp}}(M) \tag{83}$$

for $\frac{3}{4} < a < 1$, then the arguments in the previous proof cannot be applied. To see this, let us consider the case $\kappa = 0$. The function F satisfying a suitably modified version of (78) reads in this case

$$F(\tau) = \left(\frac{\tau}{\eta}\right)^{\sqrt{a}}, \qquad \tau \in [2\zeta, \eta]. \tag{84}$$

Then, by construction, the function F satisfies

$$F(\tau) = \left(\frac{2}{\eta}\right)^{\sqrt{a}} \zeta^{\sqrt{a}-1}(\tau - \zeta) \quad \text{for } [\zeta, 2\zeta]. \tag{85}$$

In particular, if $a < 1$, we cannot find a constant K independent of ζ such that $F'(\tau) = (2/\eta)^{\sqrt{a}}\zeta^{\sqrt{a}-1} \leq K$. On the other hand, for $a \geq 1$, we have $F'(\tau) = (2/\eta)^{\sqrt{a}}\zeta^{\sqrt{a}-1} \leq 2^{\sqrt{a}}/\eta^{2-\sqrt{a}} = K(\eta)$ and the previous argument works exactly in the same way.

4.2. **Proof of the criterion.**

Proof of Theorem 4.1. We divide the proof of the theorem in two steps.

Part 1: essential self-adjointness. By Proposition 4.6, the operator H is semibounded. Thus, by a well-known criterion (see [20, Thm. X.I and Corollary]), H is essentially self-adjoint if and only if there exists $E < 0$ such that the only solution of $H^*\psi = E\psi$ is $\psi \equiv 0$. This is guaranteed by the Agmon-type estimate of Proposition 4.7, whose hypotheses are satisfied again by Proposition 4.6.

Part 2: compactness of the resolvent. The proof follows the same steps as in [18, Prop. 3.7], and makes use of the sub-Riemannian version of the Rellich-Kondrachov theorem (Lemma 4.3). For the sake of completeness, we sketch here the proof.

First of all notice that it suffices to show that there exists $z \in \mathbb{R}$ such that the resolvent $(H^* - z)^{-1}$ is compact on $L^2(M)$. This follows by the first resolvent formula (see [19, Thm. VIII.2]) and by the fact that compact operators are an ideal of bounded ones. Moreover, by 4.6 H is a semibounded operator, i.e.,

$$\langle Hu, u\rangle \geq c\|u\|^2, \qquad \forall u \in C_c^\infty(M). \tag{86}$$

Hence, by [21, Thm. XIII.64] its spectrum consists of discrete eigenvalues with finite multiplicity.

Notice that (86), together with the fact that H^* is self-adjoint, imply that $(H^* - z)^{-1}$ is well defined for every $z \leq c$ and $\|(H^* - z)^{-1}\| \leq 1/(c - z)$. To prove compactness of the operator $(H^* - z)^{-1} : L^2(M) \to \mathrm{Dom}(H^*)$ for $z < c$ we need to show that for any bounded sequence $\psi_n \in L^2(M)$, say $\|\psi_n\| \leq (c - z)$, the image sequence $u_n = (H^* - z)^{-1}\psi_n \in \mathrm{Dom}(H^*)$ has a subsequence converging in $L^2(M)$. Notice that $\|u_n\| \leq 1$.

In order to extract a converging subsequence of u_n, we prove estimates for the functions u_n localized close and far away from the singular region. We provide such estimates for any function $u \in \mathrm{Dom}(H^*) \subseteq W^1_{\mathrm{loc}}(M)$ (see Lemma 4.2), setting $\psi = (H^* - z)u$, and we will then apply them to the elements of the sequence u_n, to extract a converging subsequence. To this purpose let $\chi_1, \chi_2 : [0, +\infty] \to \mathbb{R}$ be real valued Lipschitz functions such that

- $0 \leq \chi_i \leq 1$ for $i = 1, 2$;

- $\chi_1 \equiv 1$ on $[0, \eta/2]$ and $\chi_1 \equiv 0$ on $[\eta, +\infty)$;
- $\chi_2 \equiv 0$ on $[0, \eta/2]$ and $\chi_2 \equiv 1$ on $[\eta, +\infty)$;
- they interpolate linearly elsewhere.

Consider the functions $\phi_i := \chi_i \circ \delta$, which are Lipschitz w.r.t. the sub-Riemannian distance. Notice that $\phi_1 + \phi_2 = 1$. Since M is relatively compact in N, ϕ_2 is compactly supported in M, implying by (50)

$$\mathcal{E}(\phi_2 u, \phi_2 u) = \operatorname{Re}\mathcal{E}(u, \phi_2^2 u) + \langle u, |\nabla\phi_2|^2 u\rangle = \operatorname{Re}\langle H^* u, \phi_2^2 u\rangle + \langle u, |\nabla\phi_2|^2 u\rangle \tag{87}$$

$$= z\|\phi_2 u\|^2 + \operatorname{Re}\langle \psi, \phi_2^2 u\rangle + \langle u, |\nabla\phi_2|^2 u\rangle \leq z\|u\|^2 + \|\psi\|\|u\| + 4\|u\|^2\eta^{-2}, \tag{88}$$

where in the last estimate we used the fact that χ_2 is linear between $\eta/2$ and η hence $\phi_2 = \chi_2 \circ \delta$ satisfies $|\nabla\phi_2| \leq |\chi_2'||\nabla\delta| \leq 2/\eta$. We deduce the following estimate "far away" from the singular region:

$$\int_{M\setminus M_{\eta/2}} |\nabla(\phi_2 u)|^2 \, d\omega = \mathcal{E}(\phi_2 u, \phi_2 u) \leq z\|u\|^2 + \|\psi\|\|u\| + 4\|u\|^2\eta^{-2}. \tag{89}$$

We now consider the localization of u close to the metric boundary. Since H is essentially self-adjoint and $H^* = \bar{H}$, we can choose a sequence $u_k \in C_c^\infty(M)$ such that u_k converges to u in the graph norm of H^*, i.e., $\|H^*(u_k - u)\| + \|u_k - u\| \to 0$ as $k \to \infty$. We deduce an upper bound for $\phi_1 u$ in M_η from the following bounds on the elements u_k as follows. First, we use (63) to obtain

$$\int_{M_\eta} |u_k|^2 \, d\omega = \int_{M_\eta} \frac{\delta^2}{1-\delta\kappa}\frac{1-\delta\kappa}{\delta^2}|u_k|^2 \, d\omega \leq \frac{\eta^2}{1-\eta\kappa}\int_{M_\eta}\left(\frac{1}{\delta^2} - \frac{\kappa}{\delta}\right)|u_k|^2 \, d\omega \tag{90}$$

$$\leq \frac{\eta^2}{1-\eta\kappa}\left(\mathcal{E}(u_k, u_k) - c\|u_k\|^2\right) = \frac{\eta^2}{1-\eta\kappa}\left(\langle H^* u_k, u_k\rangle - c\|u_k\|^2\right). \tag{91}$$

Then, passing to the limit $k \to \infty$, and recalling that $\phi_1 \leq 1$, we get

$$\int_{M_\eta} |\phi_1 u|^2 \, d\omega \leq \int_{M_\eta} |u|^2 \, d\omega \tag{92}$$

$$\leq \frac{\eta^2}{1-\eta\kappa}\left(\langle H^* u, u\rangle - c\|u\|^2\right) = \frac{\eta^2}{1-\eta\kappa}\left((z-c)\|u\|^2 + \|\psi\|\|u\|\right). \tag{93}$$

We apply the latter construction to each element $u_n = (H^* - z)^{-1}\psi_n \in \operatorname{Dom}(H^*)$, setting $u_n = u_{n,1} + u_{n,2}$ with $u_{n,i} = \phi_{n,i} u_n$. Recalling that $\|u_n\| \leq 1$, equation (89) applied to $u = u_n$, $\psi = \psi_n$, implies

$$\|u_{n,2}\|^2_{W^1(M)} = \int_{M\setminus M_{\eta/2}} |\nabla u_{n,2}|^2 \, d\omega + \|u_{n,2}\|^2 \leq c + 4\eta^{-2} + 1. \tag{94}$$

That is, $u_{n,2}$ is bounded in $W^1(M)$. Moreover, by construction, $u_{n,2} \in W^1_{\mathrm{comp}}(\Omega)$ where $\Omega = \{\delta \geq \eta/2\} \subset M$ is a compact domain with smooth boundary by Proposition 3.1. This implies that $u_{n,2}$ converges up to subsequences in $L^2(\Omega)$ (thus in $L^2(M)$) by the sub-Riemannian Rellich-Kondrachov theorem, see Lemma 4.3.

On the other hand, (92) implies that for some constant C independent of η, we have

$$\|u_{n,1}\|^2 = \int_{M_\eta} |u_{n,1}|^2 \, d\omega \leq \frac{\eta^2}{1-\eta\kappa} 2(c-z) \leq C\eta^2, \tag{95}$$

Since η in the Hardy inequality (63) can be arbitrarily small, say $\tilde{\eta}_k^2 = 1/k$, we actually proved that for all $k \in \mathbb{N}$, there is a subsequence $n \mapsto \gamma_k(n)$ such that $u_{\gamma_k(n)} = \sum_{i=1}^2 u_{\gamma_k(n),i}$ with $\|u_{\gamma_k(n),1}\| \leq C/k$ and $u_{\gamma_k(n),2}$ convergent in $L^2(M)$. Exploiting this fact, we extract a Cauchy subsequence of u_n, yielding the compactness of $(H^* - z)^{-1}$, and concluding the proof. Details on the extraction are in [18, Prop. 3.7]. □

5. Applications to the intrinsic sub-Laplacian

The main interest of our result is in its application to the study of sub-Riemannian manifolds endowed with the intrinsic Popp's measure. More precisely, given a complete sub-Riemannian manifold N, we are interested in studying essential self-adjointness of the sub-Laplacian $\Delta = \Delta_{\mathcal{P}}$, where $\mathcal{P}$ is the Popp's measure. As discussed in Section 2.2, $\mathcal{P}$ is smooth on the equiregular region of N (the largest open set on which the sub-Riemannian structure is equiregular), and blows up on its complement: the singular region $\mathcal{Z}$. We assume that $\mathcal{Z}$ is a smooth embedded hypersurface with no characteristic points, and that $\mathcal{Z}$ is compact (or, at least, that it has strictly positive injectivity radius, see Remark 4.1). Furthermore, $\mathrm{Dom}(\Delta) = C_c^\infty(M)$ with $M = N \setminus \mathcal{Z}$ or any of its connected components.

We start by considering a family of structures generalizing the Martinet structure, which has been presented in the introduction. These are complete sub-Riemannian structures on $\mathbb{R}^3$, equiregular outside a hypersurface $\mathcal{Z} \subset \mathbb{R}^3$, on which the distance from $\mathcal{Z}$ is explicit. Using Theorem 4.1 (and Remark 4.1) we deduce essential self-adjointness of $\Delta = \Delta_{\mathcal{P}}$ defined on $C_c^\infty(N \setminus \mathcal{Z})$.

Example 5.1 (k-Martinet distribution). Let $k \in \mathbb{N}$. We consider the sub-Riemannian structure on $\mathbb{R}^3$ defined by the following global generating family of vector fields:

$$X_1 = \partial_x, \qquad X_2 = \partial_y + x^{2k}\partial_z. \tag{96}$$

The singular region is $\mathcal{Z} = \{x = 0\}$ and the distance from $\mathcal{Z}$ is $\delta(x, y, z) = |x|$. Using formula (28), the associated Popp's measure turns out to be

$$\mathcal{P} = \frac{1}{2\sqrt{2}k|x|^{2k-1}}\, dx \wedge dy \wedge dz. \tag{97}$$

The case $k = 1$ is the standard Martinet structure considered in the introduction. Notice that the injectivity radius from $\mathcal{Z}$ is infinite, hence even if $\mathcal{Z}$ is not compact we can apply Theorem 4.1. We compute the effective potential V_{eff} using (59). Indeed we have

$$\theta = \theta(x) = \frac{1}{2}\log\frac{1}{2\sqrt{2}kx^{2k-1}}, \tag{98}$$

and thus, using (59), we have

$$V_{\mathrm{eff}}(x) = \frac{4k^2 - 1}{4x^2} \geq \frac{3}{4x^2}, \qquad \forall k \geq 1. \tag{99}$$

Hence (46) is satisfied, and $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(\mathbb{R}^3 \setminus \mathcal{Z})$ is essentially self-adjoint.

The study of condition (46) is a difficult task, because it requires the explicit knowledge of the distance from the singular set. In the following we define a class of sub-Riemannian structures, to which Theorem 4.1 applies, without knowing an explicit expression for δ. Let ϖ be a reference measure, smooth and positive on the whole N and let $\mathcal{P}$ denote Popp's measure, smooth on $M = N \setminus Z$. We define the function $\rho : N \to \mathbb{R}$ by setting

$$\rho(p) = \begin{cases} \left(\frac{d\mathcal{P}}{d\varpi}\right)^{-1}(p) & \text{if } p \in N \setminus \mathcal{Z}, \\ 0 & \text{if } p \in \mathcal{Z}. \end{cases} \tag{100}$$

This is the unique continuous extension to $\mathcal{Z}$ of the reciprocal of the Radon-Nikodym derivative of $\mathcal{P}$ with respect to ϖ. Notice that ρ is smooth on $N \setminus \mathcal{Z}$.

Definition 5.1. We say that a sub-Riemannian manifold N is *Popp-regular* if it is equiregular outside a smooth embedded hypersurface $\mathcal{Z}$ containing no characteristic points, and there exists $k \in \mathbb{N}$ such that, for all $q \in \mathcal{Z}$ there exists a neighborhood $\mathcal{O}$ of q and a smooth submersion $\psi : \mathcal{O} \to \mathbb{R}$ such that the function ρ defined in (100) satisfies $\rho|_{\mathcal{O}} = \psi^k$.

Definition 5.1 generalizes the notion of *regular* almost Riemannian structure given in [18, Def. 7.10]. Notice that the sub-Riemannian structure in Example 5.1 is Popp-regular.

Proposition 5.2. *Let N be a complete and Popp-regular sub-Riemannian manifold, with compact singular set $\mathcal{Z}$. Then, the sub-Laplacian $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$ or one of its connected components. Moreover, if M is relatively compact, the unique self-adjoint extension of $\Delta_{\mathcal{P}}$ has compact resolvent.*

Proof. We start by noticing that the proof of Claim iii) in Proposition 3.1 can be modified in such a way that, for any $q \in \mathcal{Z}$, we construct local coordinates $(t,x) \in (-\varepsilon,\varepsilon) \times \mathbb{R}^{n-1}$, defined in a neighborhood $\mathcal{O} \subset N$ of q, with respect to which we have

$$\mathcal{Z} \cap \mathcal{O} = \{t = 0\}, \qquad \delta(t,x) = t, \qquad \nabla\delta(t,x) = \partial_t. \tag{101}$$

In fact, given a coordinate neighborhood $V \subseteq \mathcal{Z}$ around q we can choose $\lambda : V \to A\mathcal{Z}$ to be a smooth non-vanishing local section of the annihilator bundle $A\mathcal{Z}$ defined in (31) with constant Hamiltonian equal to $1/2$. Then, the map $E(x, t\lambda(x))$ is a smooth diffeomorphism satisfying (101), where E is defined as in (33).

Let now ϖ be a smooth measure on N and consider the function ρ defined in (100). By assumption, we have $\rho = \psi^k$, for a smooth submersion ψ. Thus, since in the coordinates just defined we have $\rho(0,x) = 0$, we must also have $\partial_t\psi(0,x) \neq 0$. This implies $\rho = t^k\phi(t,x)$ for a smooth never vanishing function. Notice that the expression of ϕ depends on the choice of the reference measure ϖ, but the fact that ϕ never vanishes does not depend on this choice. We compute the effective potential as

$$V_{\mathrm{eff}}|_{\mathcal{O}\setminus\mathcal{Z}} = \left(\frac{\Delta|t|}{2}\right)^2 + \partial_t\left(\frac{\Delta|t|}{2}\right) \tag{102}$$

$$= \frac{k(k+2)}{4t^2} + \frac{k^2}{2|t|}\frac{\partial_t\phi(t,x)}{\phi(t,x)} + \frac{k(k+2)}{4}\frac{\partial_t\phi(t,x)^2}{\phi(t,x)^2} - \frac{k}{2}\frac{\partial_t^2\phi(t,x)}{\phi(t,x)}. \tag{103}$$

Up to restricting to a smaller, compact subset $\mathcal{O}' \simeq [-\varepsilon',\varepsilon'] \times [-1,1]^{n-1}$, we get the estimate $V_{\mathrm{eff}}|_{\mathcal{O}'\setminus\mathcal{Z}} \geq 3/(4t^2) - \kappa'/|t|$ for some constant $\kappa' > 0$. By compactness of $\mathcal{Z}$, and up to choosing a sufficiently small ε, we can cover $M_\varepsilon = \{0 < \delta < \varepsilon\}$ with a finite number of coordinate neighborhoods $\mathcal{O}'$ and we obtain the global estimate $V_{\mathrm{eff}} \geq 3/(4\delta^2) - \kappa/\delta$ on M_ε. We conclude by applying Theorem 4.1. □

Remark 5.1. The compactness of $\mathcal{Z}$, used to produce uniform lower bounds for the V_{eff}, is not a necessary condition. For instance, the singular regions of Martinet-type structures of Example 5.1 are not compact. Nonetheless, the k-Martinet structures are Popp-regular and, as we have seen, Theorem 4.1 still yields the essential self-adjointness of $\Delta_{\mathcal{P}}$.

We show an example of a non-Popp-regular sub-Riemannian structure, to which Theorem 4.1 still applies, yielding the essential self-adjointness of $\Delta = \Delta_{\mathcal{P}}$.

Example 5.2 (suggested by F. Jean). Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases} \tag{104}$$

and consider the sub-Riemannian structure on $\mathbb{R}^7$ given by the global generating family:

$$X_1 = \partial_1, \qquad X_2 = \partial_2 + x_1\partial_3 + x_1^2\partial_4 + x_1^3\partial_5, \tag{105}$$

$$X_3 = \partial_6 + f(x_1)\partial_4, \qquad X_4 = \partial_7 + f(-x_1)\partial_5, \tag{106}$$

where $x = (x_1, \dots, x_7)$ denote the coordinates in $\mathbb{R}^7$ and ∂_i denotes the derivative with respect to the i-th coordinate. The singular region is the set $\mathcal{Z} = \{x_1 = 0\}$. The distribution $\mathcal{D}^2$ is generated by $\mathcal{D}^1$ and by the vector fields

$$[X_1, X_2] = \partial_3 + 2x_1\partial_4 + 3x_1^2\partial_5, \tag{107}$$

$$[X_1, X_3] = f'(x_1)\partial_4, \tag{108}$$

$$[X_1, X_4] = -f'(-x_1)\partial_5, \tag{109}$$

hence $\dim(D_x^2) = 6$ at any point $x \in \mathbb{R}^7$, $x_1 \neq 0$. In particular, $[X_1, X_3]$ is proportional to ∂_4 if $x_1 > 0$, and 0 otherwise, while $[X_1, X_4]$ is proportional to ∂_5 if $x_1 < 0$ and 0 otherwise. The distribution $\mathcal{D}^3$ is generated by $\mathcal{D}^2$ and by the vector fields

$$[X_1, [X_1, X_2]] = 2\partial_4 + 6x_1\partial_5, \tag{110}$$

$$[X_1, [X_1, X_3]] = f''(x_1)\partial_4, \tag{111}$$

$$[X_1, [X_1, X_4]] = f''(-x_1)\partial_5. \tag{112}$$

Hence $\dim(D_x^3) = 7$ if $x_1 < 0$ and $\dim(D_x^3) = 6$ if $x_1 > 0$.

We compute Popp's measure separately in the regions $R_- = \{x_1 < 0\}$ and $R_+ = \{x_1 > 0\}$. In R_- we choose the adapted basis

$$\underbrace{X_1,\ X_2,\ X_3,\ X_4}_{\mathcal{D}^1},\quad \underbrace{X_5 = [X_1, X_2],\ X_6 = [X_1, X_3] + [X_1, X_4]}_{\mathcal{D}^2/\mathcal{D}^1},\quad \underbrace{X_7 = f(-x_1)\partial_4}_{\mathcal{D}^3/\mathcal{D}^2}. \tag{113}$$

Using the explicit formula (28), we obtain:

$$\mathcal{P}|_{R_-} = \frac{1}{4|f'(-x_1)|} dx_1 \wedge \cdots \wedge dx_7 = \frac{|x_1|^3 e^{-1/x_1^2}}{8} dx_1 \wedge \cdots \wedge dx_7. \tag{114}$$

In R_+ we choose the adapted basis

$$\underbrace{X_1,\ X_2,\ X_3,\ X_4}_{\mathcal{D}^1},\quad \underbrace{X_5 = [X_1, X_2],\ X_6 = [X_1, X_3]}_{\mathcal{D}^2/\mathcal{D}^1},\quad \underbrace{X_7 = [X_1, [X_1, [X_1, X_2]]]}_{\mathcal{D}^4/\mathcal{D}^3} \tag{115}$$

and we obtain, using (28):

$$\mathcal{P}|_{R_+} = \frac{1}{12|f'(x_1)|} dx_1 \wedge \cdots \wedge dx_7 = \frac{x_1^3 e^{1/x_1^2}}{24} dx_1 \wedge \cdots \wedge dx_7. \tag{116}$$

Although the above computation shows that the function ρ defined in (100) is not a submersion, we can nevertheless compute the effective potential on both sides of the singular region, exploiting the fact that the distance from the singular region is $\delta(x_1, \ldots, x_7) = |x_1|$. (Here, the reference measure ϖ is taken to be the Lebesgue measure.) For $\mathcal{P}|_{R_+}$ we have

$$\mathcal{P}|_{R_+} = e^{2\theta} dx_1 \wedge \ldots dx_7, \qquad \text{with} \qquad \theta = \theta(x_1) = \frac{1}{2}\left(\log(x_1^3) + \frac{1}{x_1^2}\right). \tag{117}$$

Hence, $V_{\text{eff}} = (\partial_1\theta)^2 + \partial_1^2\theta \sim 1/x^6$, which is clearly greater than $3/(4x_1^2)$ in a uniform neighborhood of $\mathcal{Z} \cap R_+$, leading to essential self-adjointness of $\Delta_{\mathcal{P}}$ defined on $C_c^\infty(R_+)$. In the same way we can prove essential self-adjointness of $\Delta_{\mathcal{P}}$ defined on $C_c^\infty(R_-)$.

Remark 5.2. Notice that in this example it is not possible to define a local frame in a neighborhood of any point of $\mathcal{Z}$ that is adapted to the flag $\mathcal{D} \subset \mathcal{D}^1 \subset \mathcal{D}^2 \subset \mathcal{D}^3 \subset \mathcal{D}^4$ on both sides of the singular region.

We generalize Example 7.2 in [18], showing an example of non-Popp-regular sub-Riemannian structure to which Theorem 4.1 does not apply.

Example 5.3 (non-Popp-regular sub-Riemannian structure). Consider the sub-Riemannian structure on $\mathbb{R}^4$ given by the following generating family of vector fields:

$$X_1 = \partial_1 + x_3\partial_4, \qquad X_2 = x_1(x_1^{2\ell} + x_2^2)\partial_2, \qquad X_3 = \partial_3. \tag{118}$$

The singular region is $\mathcal{Z} = \{x_1 = 0\}$. The following set of vector fields is an adapted frame on $\mathbb{R}^4 \setminus \mathcal{Z}$.

$$\underbrace{X_1,\ X_2,\ X_3,}_{\mathcal{D}^1} \qquad \underbrace{X_4 = [X_3, X_1] = \partial_4}_{\mathcal{D}^2/\mathcal{D}^1}. \tag{119}$$

Using formula (28), we have the following expression for Popp's measure

$$\mathcal{P} = \frac{1}{\sqrt{2}x_1(x_1^{2\ell}+x_2^2)} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \tag{120}$$

or, equivalently, $\mathcal{P} = x_1^{a(x)} e^{2\varphi(x)} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, where

$$a(x) = \begin{cases} -(2\ell+1) & x_2 = 0, \\ -1 & x_2 \neq 0, \end{cases} \qquad \varphi(x) = \begin{cases} -\frac{1}{2}\log\sqrt{2} & x_2 = 0, \\ -\frac{1}{2}\log\left(\sqrt{2}(x_1^{2\ell}+x_2^2)\right) & x_2 \neq 0. \end{cases} \tag{121}$$

Noticing that $\delta(x_1, x_2, x_3, x_4) = x_1$, the effective potential reads

$$V_{\text{eff}} = \frac{a(x)(a(x)-2)}{4x_1^2} + R(x), \text{ with } R(x) = \frac{a(x)}{x_1}\partial_1\varphi(x) + (\partial_1\varphi(x))^2 + \partial_1^2\varphi(x). \tag{122}$$

We have

$$R(x) = \begin{cases} 0 & x_2 = 0, \\ \frac{\ell t^{2\ell-2}}{(t^{2\ell}+x_2^2)^2}\left[(\ell+2)t^{2\ell} + (2-2\ell)x_2^2\right] & x_2 \neq 0. \end{cases} \tag{123}$$

Combining (121)-(123) we deduce that $V_{\text{eff}} = 3/(4x_1^2) + R(x)$ if $x_2 \neq 0$, and it is easy to see that the behavior of $R(x)$ depends on the choice of the parameter ℓ. In particular, if $\ell = 1$, $R(x) \geq 0$ and we deduce essential self-adjointness of $\Delta = \Delta_{\mathcal{P}}$ by Theorem 4.1. On the other hand, if $\ell > 1$, along any sequence $x^i = (1/i, 1/i, 0, 0)$, we have $x_1^i R(x_i) \to -\infty$. Hence, we cannot apply Theorem 4.1.

Acknowledgments

This research has been supported by the Grant ANR-15-CE40-0018 of the ANR, by the iCODE institute (research project of the Idex Paris-Saclay). The first author has been partially supported by the GNAMPA Indam project "Problemi nonlocali e degeneri nello spazio euclideo". This research benefited from the support of the "FMJH Program Gaspard Monge in optimization and operation research" and from the support to this program from EDF.

References

[1] A. Agrachev, D. Barilari, and U. Boscain. Introduction to Riemannian and sub-Riemannian geometry (Lecture Notes). http://webusers.imj-prg.fr/ davide.barilari/notes.php. v11/07/15.

[2] A. Agrachev, U. Boscain, J.-P. Gauthier, and F. Rossi. The intrinsic hypoelliptic laplacian and its heat kernel on unimodular lie groups. *Journal of Functional Analysis*, 256(8):2621 – 2655, 2009.

[3] A. Agrachev, U. Boscain, and M. Sigalotti. A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds. *Discrete Contin. Dyn. Syst.*, 20(4):801–822, 2008.

[4] A. A. Agrachev and Y. L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[5] D. Barilari and L. Rizzi. A formula for Popp's volume in sub-Riemannian geometry. *Anal. Geom. Metr. Spaces*, 1:42–57, 2013.

[6] U. Boscain and C. Laurent. The Laplace-Beltrami operator in almost-Riemannian geometry. *Ann. Inst. Fourier (Grenoble)*, 63(5):1739–1770, 2013.

[7] U. Boscain and D. Prandi. Self-adjoint extensions and stochastic completeness of the Laplace-Beltrami operator on conic and anticonic surfaces. *J. Differential Equations*, 260(4):3234–3269, 2016.

[8] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[9] Y. Colin de Verdière and F. Truc. Confining quantum particles with a purely magnetic field. *Ann. Inst. Fourier (Grenoble)*, 60(7):2333–2356 (2011), 2010.

[10] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[11] B. Franchi, P. Hajł asz, and P. Koskela. Definitions of Sobolev classes on metric spaces. *Ann. Inst. Fourier (Grenoble)*, 49(6):1903–1924, 1999.

[12] B. Franchi, R. Serapioni, and F. Serra Cassano. Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields. *Boll. Un. Mat. Ital. B (7)*, 11(1):83–117, 1997.

[13] N. Garofalo and D.-M. Nhieu. Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces. *J. Anal. Math.*, 74:67–97, 1998.

[14] R. Ghezzi and F. Jean. On measures in sub-Riemannian geometry. *Actes du séminaire de Théorie spectrale et géométrie (in press)*, Feb. 2017.

[15] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[16] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[17] G. Nenciu and I. Nenciu. On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in $\mathbb{R}^n$. *Ann. Henri Poincaré*, 10(2):377–394, 2009.

[18] D. Prandi, L. Rizzi, and M. Seri. Quantum confinement on non-complete Riemannian manifolds. *J. Spectral Theory (in press)*, 2017.

[19] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[20] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[21] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[22] L. P. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137:247–320, 1976.

[23] R. S. Strichartz. Sub-Riemannian geometry. *J. Differential Geom.*, 24(2):221–263, 1986.

† Inria, team GECO & CMAP, École Polytechnique, CNRS, Université Paris-Saclay, Palaiseau, France

E-mail address: valentina.franceschi@inria.fr

♭ CNRS, Laboratoire des Signaux & Systémes, CentraleSupélec, Gif-sur-Yvette, France

E-mail address: dario.prandi@l2s.centralesupelec.fr

♯ Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France

E-mail address: luca.rizzi@univ-grenoble-alpes.fr