ON THE MEASURE AND THE STRUCTURE OF THE FREE BOUNDARY OF THE LOWER DIMENSIONAL OBSTACLE PROBLEM

MATTEO FOCARDI AND EMANUELE SPADARO

Abstract. We provide a thorough description of the free boundary for the lower dimensional obstacle problem in $\mathbb{R}^{n+1}$ up to sets of null $\mathcal{H}^{n-1}$ measure. In particular, we prove

(i) local finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary,
(ii) $\mathcal{H}^{n-1}$-rectifiability of the free boundary,
(iii) classification of the frequencies up to a set of dimension at most $(n-2)$ and classification of the blow-ups at $\mathcal{H}^{n-1}$ almost every free boundary point.

1. Introduction

Thin obstacle-type problems naturally appear in several models of applied sciences, such as contact mechanics (cf. the classical Signorini problem) and, as pointed out more recently, in free boundary problems for fractional diffusions, such as quasi-geostrophic flows, American options' pricing, anomalous diffusions etc... Due to their character of prototypical nonlinear and non-local equations, in the recent years this class of problems has been intensively studied, culminating in several important contributions and breakthroughs (cf., *e.g.*, [4, 11, 5, 12, 23, 14, 29, 7, 20, 3]). Nevertheless, many important questions are not yet answered, most importantly the ones concerning the global structure of the free boundary, which according to the available results in the literature is not excluded to have infinite measure or to be fractal, already in the simplest model cases.

Here we answer to this and to other related questions, such as the uniqueness of blow-ups and the structure of the free boundary for solutions to the thin obstacle problem, giving a complete description of the top-stratum of the free boundary up to a set of $\mathcal{H}^{n-1}$-measure zero. These results are new also in the framework of the classical Signorini problem in elasticity (for the antiplane case) and they are obtained by a combination of analytical and geometric measure theory arguments which can be suitably exploited also for similar free boundary type problems.

1.1. **The problem.** In this article we consider a class of lower dimensional obstacle problems. In order to state them, for any subset $E \subset \mathbb{R}^{n+1}$ we set

$$E^+ := E \cap \big\{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\big\} \quad \text{and} \quad E' := E \cap \big\{x_{n+1} = 0\big\}.$$

For any point $x \in \mathbb{R}^{n+1}$ we will write $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. Moreover, $B_r(x) \subset \mathbb{R}^{n+1}$ denotes the open ball centered at $x \in \mathbb{R}^{n+1}$ with radius $r > 0$, and $\overline{B_r}(x)$ its closure (we omit to write the point x if the origin). For every $R > 0$, we denote by $\mathscr{A}_R$ the set of functions in the weighted Sobolev space $H^1(B_R, |x_{n+1}|^a\mathcal{L}^{n+1})$, with $a \in (-1, 1)$, which are even symmetric with respect to x_{n+1} and which have positive traces on B'_R:

$$\mathscr{A}_R := \Big\{v \in H^1(B_R, |x_{n+1}|^a\mathcal{L}^{n+1}) : v(x', x_{n+1}) = v(x', -x_{n+1}) \quad \text{and} \quad v(x', 0) \geq 0\Big\}.$$

Key words and phrases. Thin obstacle problem, free boundary, rectifiability, blowup, uniqueness.

E. S. has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) through a Visiting Professor Fellowship. E. S. is very grateful to the DiMaI "U. Dini" of the University of Firenze for the support during the visiting period. M. F. is a member of the GNAMPA of INdAM. He warmly thanks the University of Leipzig for the support during a visit when part of this work was conceived.

The thin obstacle problems we consider are then the following:

$$\begin{cases} u(x',0) \geq 0 & \text{for } \ (x',0) \in B'_R, \\ u(x',x_{n+1}) = u(x',-x_{n+1}) & \text{for } \ x = (x',x_{n+1}) \in B_R, \\ \operatorname{div}\big(|x_{n+1}|^a \nabla u(x)\big) = 0 & \text{for } \ x \in B_R \setminus \big\{(x',0) \,:\, u(x',0) = 0\big\}, \\ \operatorname{div}\big(|x_{n+1}|^a \nabla u(x)\big) \leq 0 & \text{in the sense of distribution in } \ B_R, \\ u(x) = g(x) & \text{for } \ x \in \partial B_R, \end{cases} \tag{1.1}$$

where $g \in \mathscr{A}_R$ is a given boundary value datum. Note that (1.1) are the Euler–Lagrange equations satisfied by the unique minimizer of the energy

$$\int_{B_R} |\nabla v|^2 |x_{n+1}|^a \mathrm{d}x$$

in the class $\mathscr{A}_{R,g} := \mathscr{A}_R \cap \big\{g + H^1_0(B_R, |x_{n+1}|^a \mathcal{L}^{n+1})\big\}$. In particular, in case $a = 0$ problem (1.1) corresponds to the well-known scalar Signorini problem. We denote by $\Lambda(u)$ the *coincidence set* of a solution u,

$$\Lambda(u) := \big\{(x',0) \in B'_R \ : u(x',0) = 0\big\},$$

and by $\Gamma(u)$ its *free boundary*, which is the topological boundary of $\Lambda(u)$ in the relative topology of B'_R. In order to avoid unnecessary complications, in this work we consider the case of zero obstacle prescribed on flat hypersurfaces only. Nevertheless, the techniques developed in the paper can be generalized to consider non-constant and non-flat obstacles, as well as for other free boundary problems (such as the fractional obstacle problem, for which the analogous results of this paper are going to appear in a future work). Moreover, we set

$$s := \frac{1-a}{2}$$

throughout the whole paper.

1.2. **A short survey of the existing literature.** In the last years there has been an intensive research activity in trying to set up the regularity properties of the solutions to (1.1) and the corresponding free boundaries. We resume in what follows the state of the art for what concerns the zero obstacle case. To this aim we introduce the following notation for the rescalings of a solution u: for every $x_0 \in \Gamma(u)$ and $r > 0$, we set

$$\bar{u}_{x_0,r}(y) := \frac{r^{\frac{n+a}{2}} u(x_0 + r\,y)}{\Big(\int_{\partial B_r} u^2 \,|x_{n+1}|^a \,\mathrm{d}\mathcal{H}^n\Big)^{1/2}} \qquad \forall\, y \in B_{\frac{R-|x_0|}{r}}. \tag{1.2}$$

By [12, Section 6] the collection of functions $\{\bar{u}_{x_0,r}\}_{r>0}$ is pre-compact in the weighted Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, |x_{n+1}|^a\,\mathcal{L}^{n+1})$. Their limiting points are called *blow-ups* of u at x_0 and are homogeneous functions, whose homogeneity depends only on x_0 and not on the extracted subsequence (for a proof see also Corollary 2.10 and Remark 2.14 below). The set of all blow-ups of a solution u at x_0 is denoted by $\mathrm{BU}(x_0)$, and their common homogeneity $\lambda(x_0)$ is called the *infinitesimal homogeneity* or the *frequency* of u at x_0 (this is indeed the limiting value, as the radius vanishes, of an Almgren's type frequency function).

The following statements summarize several results available in the current literature.

A. Optimal regularity of u**.** The solutions u to (1.1) are one-sided $C^{1,s}$, $s = {}^{(1-a)}\!/_2$. More precisely, $u \in \mathrm{Lip}(B_1) \cap C^{1,s}(B_1^{\pm} \cup B'_1)$, as proved by Athanasopoulos and Caffarelli [4] for $a = 0$, and by Caffarelli, Salsa and Silvestre [12] for all $a \in (-1,1)$ (see also [21, 22, 9, 28, 36, 37, 35] for previous results).

B. Free boundary regularity. The free boundary $\Gamma(u)$ can be split as:

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Sing}(u) \cup \mathrm{Other}(u), \tag{1.3}$$

with these subsets being pairwise disjoint, and more precisely

(i) $\mathrm{Reg}(u)$ is the subset of points in $\Gamma(u)$ in which blow-ups are $(1+s)$-homogeneous. $\mathrm{Reg}(u)$ is relatively open in $\Gamma(u)$ and it is an analytic $(n-1)$-dimensional submanifold of $\mathbb{R}^{n+1}$ (the $C^{1,\alpha}$ regularity has been shown in [5, 12] – see also [20, 25] for a different proof based on the epiperimetric inequality; higher regularity follows from [14, 29]);

(ii) $\mathrm{Sing}(u)$ is the subset of points in $\Gamma(u)$ for which the blow-ups are $2m$-homogeneous. In the case of the Signorini problem $a = 0$, they are also characterized by the fact that their contact sets have density zero with respect to $\mathcal{H}^n$. Furthermore, in such a case Garofalo and Petrosyan [23] proved that $\mathrm{Sing}(u)$ is contained in a countable union of C^1-regular $(n-1)$-dimensional submanifolds.

C. Blow-up analysis. The blow-ups of u at a free boundary point x_0 satisfy the ensuing properties:

(i) $\mathrm{BU}(x_0) \subseteq \mathcal{H}_{\lambda(x_0)}$, the latter set being the positive cone of $\lambda(x_0)$-homogeneous local solutions to (1.1) even with respect to x_{n+1}. Moreover, the possible values of the frequency $\lambda(x_0)$ lie in the set $\{1+s\} \times [2, +\infty)$ (cf. [12]);

(ii) the blow-ups are unique both at every point of $\mathrm{Reg}(u)$ (cf. [12]), and at every point of $\mathrm{Sing}(u)$ for the Signorini problem $a = 0$ (cf. [23]).

Despite these significant achievements, many issues on the analysis of the regularity of the free boundary and the corresponding blow-ups of solutions to (1.1) remain still unsolved, even for the scalar Signorini problem. The most striking fact is that nothing is known about the global nature of the free boundary, which in principle is not known to have the right dimensionality of a boundary in $\mathbb{R}^n \times \{0\}$ (*i.e.*, $n-1$), nor it is known to retain any boundary-like structure (as far as we know, $\Gamma(u)$ can be even fractal). In particular, there are no results about the subset of free boundary points $\mathrm{Other}(u)$, which are neither regular nor singular (according to the definitions in literature). On the other hand, explicit examples show that $\mathrm{Other}(u)$ is in general not empty, and indeed it may coincide with the full free boundary (cf. § 8)!

1.3. **The main results of the paper.** In this paper we answer to some of the questions mentioned above, such as that concerning the dimension of the free boundary, and we give a comprehensive description of the set $\mathrm{Other}(u)$ and $\mathrm{Sing}(u)$ in the general case $a \in (-1, 1)$ up to a null $\mathcal{H}^{n-1}$ set. Our results are already new for the case of the Signorini problem $a = 0$ and extend in various directions what was previously known. In particular, the short outcome of our analysis is the global picture of the free boundary of the thin obstacle problem as an $(n-1)$-dimensional set with locally finite measure (in fact with finite Minkowski content) satisfying almost everywhere a similar stratification as for the classical obstacle problem (including some uniqueness results of the blow-ups), cf. [8, 10, 38, 30].

We start off showing that the free boundary is $(n-1)$-dimensional in a strong measure theoretic sense.

Theorem 1.1. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then, the free boundary $\Gamma(u)$ has locally finite $(n-1)$-dimensional Minkowski content: i.e., for every $K \subset\subset B_1'$ there exists a constant $C(K) > 0$ such that*

$$\mathcal{L}^{n+1}\big(\mathcal{T}_r(\Gamma(u) \cap K)\big) \leq C(K)\, r^2 \quad \forall\, r \in (0,1), \tag{1.4}$$

where $\mathcal{T}_r(E) := \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, E) < r\}$ for all $E \subseteq \mathbb{R}^n$.

Next, we prove the following geometric regularity result for the free boundary establishing its $\mathcal{H}^{n-1}$-rectifiability.

Theorem 1.2. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then, there exist at most countably many C^1-regular submanifolds M_i of dimension $n-1$ in $\mathbb{R}^{n+1}$ such that*

$$\mathcal{H}^{n-1}\Big(\Gamma(u) \setminus \bigcup_{i \in \mathbb{N}} M_i\Big) = 0. \tag{1.5}$$

The last result concerns one of the major open question in the field, namely to determine the possible values of the frequency $\lambda(x_0)$, or equivalently the smallest set $J \subset (0, \infty)$ for which $\mathrm{BU}(x_0) \subseteq \cup_{\lambda \in J} \mathcal{H}_\lambda$ for every $x_0 \in \Gamma(u)$, recall that $\mathcal{H}_\lambda$ denotes the set of λ-homogeneous solutions to (1.1). As explained in C. (i) above, it is known that

$$\{2m, 2m-1+s\}_{m \in \mathbb{N} \setminus \{0\}} \subseteq J \subseteq \{1+s\} \times [2, \infty).$$

Moreover, by definition $\lambda(x_0) = 1+s$ for all $x_0 \in \mathrm{Reg}(u)$ and $\lambda(x_0) \in 2\mathbb{N} \setminus \{0\}$ for all $x_0 \in \mathrm{Sing}(u)$. In the following theorem we make a step forward to clarify this stage.

Theorem 1.3. *Let u be a solution to the lower dimensional obstacle problem (1.1) in B_1. Then, there exists a subset $\Sigma(u) \subset \Gamma(u)$ with Hausdorff dimension at most $n-2$ such that*

$$\lambda(x_0) \in \{2m, 2m-1+s, 2m+2s\}_{m \in \mathbb{N} \setminus \{0\}} \quad \forall\, x_0 \in \Gamma(u) \setminus \Sigma(u).$$

In addition, for $\mathcal{H}^{n-1}$-a.e. point $x_0 \in \Gamma(u) \setminus \Sigma(u)$ with frequency $\lambda(x_0) \in \{2m, 2m-1+s\}_{m \in \mathbb{N} \setminus \{0\}}$ the blow-up of u at x_0 is unique and depends on two variables only: namely,

$$\mathrm{BU}(x_0) = \Big\{ \bar{h}_{\lambda(x_0)}(x \cdot e_{x_0}, x_{n+1}) \Big\},$$

for some $e_{x_0} \in \mathbb{R}^{n+1}$ with $|e_{x_0}| = 1$ and $e_{x_0} \cdot e_{n+1} = 0$, and $\bar{h}_{\lambda(x_0)}$ uniquely determined by $\lambda(x_0)$.

1.4. **Comments on the main results.** A few remarks are in order.

1.4.1. *Finite measure.* The main consequence of Theorem 1.1 is that the free boundary has locally finite $\mathcal{H}^{n-1}$ measure:

$$\mathcal{H}^{n-1}\big(\Gamma(u) \cap K\big) < +\infty \quad \forall\, K \subset\subset \mathbb{R}^n. \tag{1.6}$$

Nevertheless, the estimate on the Minkowski content is significantly stronger: among the other consequences, (1.4) implies, for instance, that the free boundary is nowhere dense. In addition, Theorem 1.2 establishes that the free boundary is a $\mathcal{H}^{n-1}$-rectifiable set, a piece of information which cannot be deduced nor implies the estimate on the Hausdorff measure (1.6).

The estimate on the Hausdorff dimension of the free boundary can be deduced independently from Theorem 1.1 and Theorem 1.2 by a different and more direct stratification argument (cf. Theorem 8.1).

1.4.2. *Structure of the free boundary.* Theorem 1.2 extends the analysis of the structure of the free boundary points to a subset of full measure of $\mathrm{Sing}(u) \cup \mathrm{Other}(u)$. Note that the structure of the points in $\mathrm{Sing}(u)$ for $a \neq 0$ had not been dealt with before in the literature. Nevertheless, Theorem 1.2 does not imply the pointwise results in B. (i) & (ii) for $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$, for the latter set if $a = 0$, because we prove a measure theoretic regularity property for $\Gamma(u)$, namely its $\mathcal{H}^{n-1}$-rectifiability (cf. (1.5)).

1.4.3. *Frequency.* Points with frequencies $2m-1+s$ and $2m+2s$, with $m \in \mathbb{N} \setminus \{0,1\}$, belong to $\mathrm{Other}(u)$, though it is not known whether they do exhaust such a set or not in general. In other words, the problem of classifying all possible frequencies for free boundary points is settled by Theorem 1.3 only up to sets of dimension at most $n-2$, but it remains open pointwise.

Moreover, if on one hand there are examples of free boundary points with frequency $2m$ and $2m-1+s$, on the other hand there are no examples of points with frequency $2m+2s$. In dimension $n=1$ one can show that such points do not exist, that is $\mathrm{Other}(u) = \{2m-1+s\}_{m \in \mathbb{N} \setminus \{0,1\}}$ and also that $\Sigma(u) = \emptyset$ (see § 8 – the case $a=0$ has been discussed in [23]). In higher dimensions it is then natural to conjecture the same results.

The reason why we are unable to rule out points with frequencies $2m+2s$ if $n \geq 2$ is related to the existence of $(2m+2s)$-homogeneous solutions with contact set $\Lambda = \mathbb{R}^n \times \{0\}$, which potentially could arise as blow-ups in a free boundary point (with the free boundary disappearing in the limit). This possibility might seem an apparent and striking discrepancy with the measure estimate and the structure result of Theorems 1.1 and 1.2 and make these results in some sense surprising (see § 8.3 for further comments).

Finally, the estimate on the Hausdorff dimension of $\Sigma(u)$ follows from its inclusion in the subset of points of $\Gamma(u)$ whose blow-ups have at most $(n-2)$ directions of invariance, for such a set the dimensional estimate is actually sharp (cf. Theorem 8.1).

1.4.4. *Blow-ups.* The uniqueness of blow-ups provided by Theorem 1.3 at points of the free boundary with frequency $2m$ and $2m-1+s$ univocally describes the infinitesimal behaviour of the solution u. In particular, it shows that the solutions look locally like a homogeneous function of a single horizontal variable and of x_{n+1}. Note that, for any different choice of the renormalization of the rescalings (1.2), either the limit does not exist or it is a multiple of $\bar{h}_\lambda$, thus justifying the notion of unique limiting profile.

For the prospective points with frequency $2m+2s$, as a by-product of the results in Appendix A, we are also able to classify all possible blow-ups.

1.5. **Concerning the proofs.** Our analysis is based on geometric measure theory techniques, which exploit and develop some ideas recently introduced in the context of minimal surfaces theory. The point of view we adopt is new in the theory of free boundaries and we believe it has many potentialities for other related problems.

The main ingredients of our study are (a variant of) Almgren's frequency function and the Peter Jones' number $\beta_\mu^{(n-1)}$ pertaining to a suitable measure μ supported on the free boundary $\Gamma(u)$ (the terminology *mean flatness* is also adopted in literature to term the β-numbers, since they provide an integral control of the flatness of the support of the underlying measure μ, see [2]). The starting point is a striking observation, which has been recently made by Naber and Valtorta [31, 32] in the context of minimal surfaces and harmonic maps theory: the square power of the mean flatness can be controlled by an average of the oscillations of a monotone density. Indeed, when this happens, a careful covering argument [31, 32], and recently developed rectifiability criteria by David–Toro [13], Azzam–Tolsa [6] and Naber–Valtorta [31, 32] lead to the local finiteness and the rectifiability of the singular sets of minimal surfaces and harmonic maps (see also the paper by De Lellis, Marchese, Spadaro and Valtorta [16] for an extension to a special case in higher co-dimension).

For our analysis of the thin obstacle problem, we generalize and develop these approaches. The starting point is an estimate of the mean flatness with respect to a Borel measures μ supported on the free boundary with the spatial oscillation of Almgren's frequency function (cf. Proposition 4.2). Note that the case of the frequency function is different from the mass ratio of a minimal surface, because the renormalization factor is intrinsically defined by the solution itself (usually a variant of the L^2-norm at the boundary of a ball), instead of being purely dimensional. This requires a novel estimate for the frequency of the solutions to the lower dimensional obstacle problem, which is based on a different set of spatial variations and is proven in Proposition 3.3: here we follow closely ideas of [16], where an analogous estimate is proved in the context of multiple-valued functions as a result of this spatial variations of the frequency.

A careful analysis of the rigidity properties of homogeneous solutions to the thin obstacle problem (1.1) (cp. Proposition 5.6) is then necessary for our argument. To this aim, as in general no growth estimate from below for solutions from the free boundary are at disposal, it is mandatory for us to introduce the set of nodal points.

Using such rigidity results and the mentioned estimate on the mean flatness via the frequency we use the covering argument and the discrete Reifenberg theorem by Naber and Valtorta in [31, 32] in order to infer Theorem 1.1. Then, Theorem 1.2 is obtained by means of the rectifiability criterion recently established by Azzam and Tolsa [6] and indipendently by Naber and Valtorta in [31, 32], while Theorem 1.3 is a consequence of Almgren's stratification principle (see, *e.g.*, [19]) and the classification of homogeneous solutions of the PDE (1.1) given in § 8 and § A.

1.6. **Structure of the paper.** We start off introducing several preliminaries in § 2. More precisely, in § 2.1 we collect the results concerning the regularity of the solutions to the thin obstacle problem. In § 2.2 we introduce the variant of the frequency function we are going to use and we derive several useful properties. We then show in § 3 how to deduce from these an oscillation estimate of the frequency. The aforementioned control of the flatness of the free boundary (defined in terms of the Peter Jones' numbers), with the oscillation of the frequency is established in Proposition 4.2. Next, § 5 is devoted to the classification results for homogeneous solutions to the PDE in (1.1) under several conditions and to study the rigidity properties of almost homogeneous solutions. Full proofs of the classification are provided in the Appendix A.

We then proceed with the proofs of Theorems 1.1, 1.2 and 1.3 in § 6, § 7 and § 8, respectively. In the corresponding section we recall the analytical results that we exploit in the proofs, namely the discrete Reifenberg theorem by Naber and Valtorta [31, 32], the rectifiability criterion by Azzam and Tolsa [6], and Almgren's stratification principle following the abstract version provided in our paper in collaboration with Marchese [19].

2. Preliminaries on the thin obstacle problem

In this section we recall some of the known results on the thin obstacle problem.

2.1. **Optimal regularity.** The following is the main existence and regularity theorem by Caffarelli, Salsa and Silvestre [12].

Theorem 2.1. *For every $g \in \mathscr{A}_1$, there exists a unique solution u to the thin obstacle problem (1.1) in B_1. Moreover, $\partial_{x_i} u \in C^s(B_1)$ for $i = 1, \dots, n$, $|x_{n+1}|^a \partial_{x_{n+1}} u \in C^\alpha(B_1)$, $0 < \alpha < 1 - s$, and there exists a constant $C_{2.1} > 0$ such that*

$$\|\nabla_\tau u\|_{C^s(B_{1/2})} + \|\mathrm{sign}(x_{n+1})\, |x_{n+1}|^a \partial_{x_{n+1}} u\|_{C^\alpha(B_{1/2})} \le C_{2.1} \, \|u\|_{L^2(B_1, |x_{n+1}|^a \mathcal{L}^{n+1})}, \tag{2.1}$$

where $\nabla_\tau u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ is the horizontal gradient.

Remark 2.2. The estimates in [12, Proposition 4.3] are given in terms of the C^0 norm of u on the right hand side of the inequality. Nevertheless, $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ satisfy $\mathrm{div}(|x_{n+1}|^a \nabla u^\pm(x)) \ge 0$ in $\mathcal{D}'(B_1')$. Therefore, by the L^∞-estimate in [17, Theorem 2.3.1] we have that

$$\|u^+\|_{C^0(B_{3/4})} + \|u^-\|_{C^0(B_{3/4})} \le C \, \|u\|_{L^2(B_1, |x_{n+1}|^a \mathcal{L}^{n+1})}, \tag{2.2}$$

and then (2.1) follows by combining [12, Proposition 4.3] and (2.2).

Remark 2.3. For later purposes we also need the estimate

$$\sup_{x \in B_\rho} |\nabla u(x) \cdot x| \le C \, \rho^{-\frac{n+1+a}{2}} \|u\|_{L^2(B_{2\rho}, |x_{n+1}|^a \mathcal{L}^{n+1})}, \tag{2.3}$$

which follows straightforwardly from (2.1).

In particular, the function u is analytic in $\big\{x_{n+1} > 0\big\} \cap B_1$ (see, *e.g.*, [26]) and the following boundary conditions holds.

Corollary 2.4. *Let u be a solution to the thin obstacle problem (1.1) in B_1. Then,*

$$\lim_{x_{n+1} \downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) = 0 \quad \text{for } \ (x', 0) \in B_1' \text{ with } \ u(x', 0) > 0, \tag{2.4}$$

$$\lim_{x_{n+1} \downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) \le 0 \quad \text{for } \ (x', 0) \in B_1', \tag{2.5}$$

$$u(x) \, \mathrm{div}(|x_{n+1}|^a \nabla u(x)) = 0 \quad \text{in } \ \mathscr{D}'(B_1). \tag{2.6}$$

Proof. Set for simplicity $f(x', 0) := \lim_{x_{n+1} \downarrow 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1})$, and note that by Theorem 2.1 we have that $f \in C^\alpha(B_1')$, $0 < \alpha < 1 - s$. By the even symmetry of u, for every $\varphi \in C^1_c(B_1)$ even symmetric we get the following: let $T := -\mathrm{div}(|x_{n+1}|^a \nabla u(x))$ in $\mathscr{D}'(B_1)$, then

$$\begin{aligned} T(\varphi(x)) &= 2 \int_{B_1^+} \nabla u(x) \cdot \nabla \varphi(x) \, |x_{n+1}|^a \, \mathrm{d}x \\ &= 2 \lim_{\epsilon \downarrow 0} \int_{\{x_{n+1} \ge \epsilon\} \cap B_1} \nabla u(x) \cdot \nabla \varphi(x) \, |x_{n+1}|^a \, \mathrm{d}x \\ &\overset{(1.1)}{=} -2 \lim_{\epsilon \downarrow 0} \int_{\{x_{n+1} = \epsilon\} \cap B_1} \partial_{n+1} u(x', \epsilon) \, \varphi(x', \epsilon) \, \epsilon^a \mathrm{d}x' \\ &= -2 \int_{B_1'} f(x', 0) \, \varphi(x', 0) \, \mathrm{d}x'. \end{aligned}$$

This shows that

$$\mathrm{div}(|x_{n+1}|^a \nabla u(x)) = -2 \, f(x', 0) \, \mathcal{H}^n \llcorner B_1'.$$

Thus, (2.4) and (2.5) follow directly from (1.1). Moreover, $u(x)\,\mathrm{div}(|x_{n+1}|^a\nabla u(x))$ is well-defined as a measure, and using (1.1) we also infer (2.6), because $f(x',0)\,u(x',0) = 0$ for all $(x',0) \in B_1'$. □

2.2. **The frequency function.** As firstly noticed by Athanasopoulos, Caffarelli and Salsa in [5], one of the main quantities which are relevant to the analysis of the solutions to the thin obstacle problem is *Almgren's frequency function*. Several variants of the frequency function have been introduced in the literature. For our purposes, we use the analog of that introduced in [15] in the context of higher co-dimension minimal surfaces.

Let $\phi : [0,+\infty) \to [0,+\infty)$ be the function given by

$$\phi(t) := \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}, \\ 2\,(1-t) & \text{for } \frac{1}{2} < t \le 1, \\ 0 & \text{for } 1 < t. \end{cases}$$

We define the frequency of a solution u to (1.1) at a point $x_0 \in B_R'$ by

$$I_u(x_0,r) := \frac{r D_u(x_0,r)}{H_u(x_0,r)} \quad \forall\, r < R - |x_0|,$$

where

$$D_u(x_0,r) := \int \phi\big(\tfrac{|x-x_0|}{r}\big)\,|\nabla u(x)|^2\,|x_{n+1}|^a \mathrm{d}x,$$

and

$$H_u(x_0,r) := -\int \phi'\Big(\tfrac{|x-x_0|}{r}\Big)\,\tfrac{u^2(x)}{|x-x_0|}\,|x_{n+1}|^a\,\mathrm{d}x.$$

Note that the frequency is well-defined as long as $H_u(x_0,r) > 0$. As $H_u(x_0,r) = 0$ implies $u \equiv 0$ by the analyticity of u in $B_R \setminus B_R'$, we infer then that the frequency is always well-defined for non-trivial solutions u. For later convenience, we introduce also the notation

$$E_u(x_0,r) := \int -\phi'\Big(\tfrac{|x-x_0|}{r}\Big)\,\tfrac{|x-x_0|}{r^2}\,\Big(\nabla u(x)\cdot\tfrac{x-x_0}{|x-x_0|}\Big)^2\,|x_{n+1}|^a\,\mathrm{d}x.$$

In what follows, when $x_0 = 0$ we shall omit to write the base point x_0 in the notation of I_u, D_u, H_u an E_u.

Remark 2.5. The principal advantage of the frequency function $I_u(x_0,r)$ is that it retains some average information of the solution u on the annulus $B_r \setminus B_{r/2}(x_0)$, whereas the classical Almgren's frequency function only involves the L^2 norm of u on the sphere $\partial B_r(x_0)$.

Remark 2.6. If u is a solution to the thin obstacle problem in B_R, then for every $r \in (0, R-|x_0|)$, $x_0 \in B_R'$ and for every $c > 0$, the function $v : B_1 \to \mathbb{R}$

$$v(y) := c\,u(x_0 + r\,y)$$

solves (1.1) in B_1 with respect to its own boundary conditions. Moreover, $I_v(0,\rho) = I_u(x_0,\rho\,r)$ for every $\rho \in (0,1)$. This shows that the frequency function is scaling invariant, and in the sequel we will use this property repeatedly.

2.3. **Monotonicity of the frequency.** The following is a simple variant of the well-known monotonicity of the frequency (cf. [11]).

Proposition 2.7. *Let u be a solution to the thin obstacle problem (1.1) in B_R. Then, for all $x_0 \in B_R'$, the map $(0, R-|x_0|) \ni r \mapsto I_u(x_0,r)$ is nondecreasing and*

$$I_u(x_0,r_1) - I_u(x_0,r_0) = \int_{r_0}^{r_1} \frac{2\,t}{H_u^2(x_0,t)}\Big(H_u(x_0,t)\;E_u(x_0,t) - D_u^2(x_0,t)\Big)\,\mathrm{d}t \tag{2.7}$$

for $0 < r_0 < r_1 < R - |x_0|$. Moreover, $I_u(x_0,\cdot) = \kappa$ for every $t \in (r_0,r_1)$ if and only if u is κ-homogeneous with respect to x_0.

Proof. We start off collecting some useful identities:

$$D_u(x_0,t) = -\frac{1}{t}\int \phi'\big(\tfrac{|x-x_0|}{t}\big)\, u(x)\nabla u(x)\cdot \tfrac{x-x_0}{|x-x_0|}\,|x_{n+1}|^a\,\mathrm{d}x, \tag{2.8}$$

$$H_u'(x_0,t) := \frac{\mathrm{d}}{\mathrm{d}t}H_u(x_0,t) = \frac{n+a}{t}\,H_u(x_0,t) + 2\,D_u(x_0,t), \tag{2.9}$$

$$D_u'(x_0,t) := \frac{\mathrm{d}}{\mathrm{d}t}D_u(x_0,t) = \frac{n+a-1}{t}\,D_u(x_0,t) + 2\,E_u(x_0,t). \tag{2.10}$$

To show (2.8), (2.9) and (2.10), we assume without loss of generality that $x_0 = 0$. For (2.8) we consider the vector field $V(x) := \phi\big(\frac{|x|}{t}\big)\,u(x)\,|x_{n+1}|^a\,\nabla u(x)$. Clearly V has compact support, and $V \in C^\infty(\mathbb{R}^{n+1}\setminus B_1', \mathbb{R}^n)$ by Theorem 2.1. Moreover, for $x_{n+1}\neq 0$

$$V(x)\cdot e_{n+1} = \phi\big(\frac{|x|}{t}\big)\,u(x)\,|x_{n+1}|^a\,\partial_{n+1}u(x)\,,$$

thus, $\lim_{y\downarrow(x',0)} V(y)\cdot e_{n+1} = 0$. Indeed, if $(x',0)\in\Lambda_\varphi(u)$ it suffices to take into account the one-sided $C^{1,\alpha}$ regularity of u in Theorem 2.1 to conclude

$$\lim_{y\downarrow(x',0)} u(y)|y_{n+1}|^a\partial_{n+1}u(x) = 0.$$

Instead, if $(x',0)\notin\Lambda_\varphi(u)$ we use (2.4) in Corollary 2.4. Thus, the distributional divergence of V is the L^1 function given by

$$\operatorname{div} V(x) = \phi\big(\tfrac{|x|}{t}\big)|\nabla u(x)|^2\,|x_{n+1}|^a + \phi'\big(\tfrac{|x|}{t}\big)u(x)\nabla u(x)\cdot\tfrac{x}{t\,|x|}\,|x_{n+1}|^a.$$

Therefore, (2.8) follows from the divergence theorem by taking into account that V is compactly supported.

Next (2.9) is a consequence of (2.8) and the direct computation

$$\begin{aligned} H_u'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-t^{n+a}\int \phi'(|y|)\,\frac{u^2(t\,y)}{|y|}\,|y_{n+1}|^a\,\mathrm{d}y\right)\\ &= \frac{n+a}{t}\,H_u(t) - 2\,t^{n+a}\int \phi'(|y|)\,u(t\,y)\,\nabla u(t\,y)\cdot\frac{y}{|y|}\,|y_{n+1}|^a\,\mathrm{d}y\\ &\overset{(2.8)}{=} \frac{n+a}{t}\,H_u(t) + 2\,D_u(t). \end{aligned}$$

Finally, to prove (2.10) we consider the vector field

$$W(x) = \left(\frac{|\nabla u|^2}{2}x - (\nabla u\cdot x)\nabla u\right)\phi\Big(\tfrac{|x|}{t}\Big)|x_{n+1}|^a.$$

By Theorem 2.1 we have that $W \in C^0_c(B_1,\mathbb{R}^n)\cap C^\infty(B_1\setminus B_1',\mathbb{R}^n)$. Moreover, Corollary 2.4 implies that $W(x',0)\cdot e_{n+1} = 0$ for all $(x',0)\in B_1'$. Thus $\operatorname{div} W$ has no singular part in B_1', and we can compute pointwise

$$\operatorname{div} W(x) = \phi'\big(\tfrac{|x|}{t}\big)\cdot\tfrac{x}{t\,|x|}\Big(\tfrac{|\nabla u|^2}{2}x - (\nabla u\cdot x)\nabla u\Big)|x_{n+1}|^a + \phi\big(\tfrac{|x|}{t}\big)\,\tfrac{n+a-1}{2}\,|\nabla u(x)|^2|x_{n+1}|^a.$$

Therefore, we infer that

$$0 = \int \operatorname{div} W(x)\,\mathrm{d}x = \int \phi'\Big(\tfrac{|x|}{t}\Big)\tfrac{|x|}{2\,t}\,|\nabla u(x)|^2|x_{n+1}|^a\,\mathrm{d}x + t\,E_u(t) + \tfrac{n+a-1}{2}\,D_u(t),$$

and we conclude (2.10) by direct differentiation

$$D_u'(t) = -\int \phi'\Big(\tfrac{|x|}{t}\Big)\tfrac{|x|}{t^2}\,|\nabla u(x)|^2|x_{n+1}|^a\,\mathrm{d}x.$$

By collecting (2.9) and (2.10), we finally compute the derivative of $\log I_u(t)$:

$$\frac{I_u'(t)}{I_u(t)} = \frac{1}{t} + \frac{D_u'(t)}{D_u(t)} - \frac{H_u'(t)}{H_u(t)} = 2\,\frac{E_u(t)}{D_u(t)} - 2\frac{D_u(t)}{H_u(t)}.$$

In particular, identity (2.7) follows at once by multiplying by $I_u(t)$ and by integrating over (r_0, r_1). In addition, by the Cauchy–Schwarz inequality, $r \mapsto I_u(r)$ is non-decreasing. Finally, if $I_u(t) = k$ for every $t \in (r_0, r_1)$, then

$$H_u(t)\, E_u(t) = D_u^2(t) \quad \forall\, t \in (r_0, r_1).$$

In particular, by the equality case in the Cauchy–Schwarz inequality, we deduce that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\nabla u(x) \cdot x = \lambda\, u(x) \quad \forall\, x \in B_{r_1} \setminus B_{r_0/2},$$

i.e. $u(x) = |x|^\lambda u(x/|x|)$ for all $x \in B_{r_1} \setminus B_{r_0/2}$. It then follows that $\lambda = k$ and by analyticity we conclude that u is k-homogeneous in the whole B_R. □

From the monotonicity of the frequency, we infer the following consequences.

Corollary 2.8. *Let u be a solution to the thin obstacle problem (1.1) in B_R. Then, for all $x_0 \in B'_R$ and $0 < r_0 < r_1 < R - |x_0|$, we have*

$$\frac{H_u(x_0, r_1)}{r_1^{n+a}} = \frac{H_u(x_0, r_0)}{r_0^{n+a}}\, e^{2\int_{r_0}^{r_1} \frac{I_u(x_0,t)}{t}\mathrm{d}t}. \tag{2.11}$$

In particular, if $A_1 \le I(x_0, t) \le A_2$ for every $t \in (r_0, r_1)$, then

$$(r_0, r_1) \ni r \mapsto \frac{H_u(x_0, r)}{r^{n+a+2A_2}} \quad \textit{is monotone decreasing,} \tag{2.12}$$

$$(r_0, r_1) \ni r \mapsto \frac{H_u(x_0, r)}{r^{n+a+2A_1}} \quad \textit{is monotone increasing.} \tag{2.13}$$

Moreover,

$$\int_{B_r(x_0)} |u|^2 |x_{n+1}|^a \,\mathrm{d}x \le r\, H_u(x_0, r). \tag{2.14}$$

Proof. The proof of (2.11) (and hence of (2.12) and (2.13)) follows from the differential equation (2.9). The proof of (2.14) is now a direct consequence:

$$\begin{aligned}\int_{B_r(x_0)} |u|^2 |x_{n+1}|^a \,\mathrm{d}x &= \sum_{k\in\mathbb{N}} \int_{B_{r/2^k} \setminus B_{r/2^{k+1}}(x_0)} |u|^2 |x_{n+1}|^a \,\mathrm{d}x \\ &\le \sum_{k\in\mathbb{N}} \frac{r}{2^k}\, H_u\big(x_0, r/2^k\big) \le r\, H_u(x_0, r),\end{aligned}$$

where in the last inequality we used that $H_u(x_0, s) \le H_u(x_0, r)$ for $s \le r$ by (2.13). □

2.4. **Lower bound on the frequency and compactness.** We first show that the frequency of a solution to (1.1) at free boundary points is bounded from below by a universal constant.

Lemma 2.9. *There exists a dimensional constant $C_{2.9} > 0$ such that, for every solution u to the thin obstacle problem (1.1) in B_R and for every $x_0 \in \Gamma(u)$, we have*

$$I_u(x_0, r) \ge C_{2.9} \quad \forall\, r \in (0, R - |x_0|). \tag{2.15}$$

Proof. By the co-area formula for Lipschitz functions we check that

$$H_u(x_0, r) = 2 \int_{\frac{r}{2}}^{r} \frac{dt}{t} \int_{\partial B_t(x_0)} |u(x)|^2\, |x_{n+1}|^a \,\mathrm{d}\mathcal{H}^n(x), \tag{2.16}$$

and

$$\begin{aligned}D_u(x_0, r) = &\int_0^{\frac{r}{2}} dt \int_{\partial B_t(x_0)} |\nabla u(x)|^2 |x_{n+1}|^a \,\mathrm{d}\mathcal{H}^n(x) \\ &+ \frac{2}{r} \int_{\frac{r}{2}}^{r} dt \int_{\partial B_t(x_0)} (r-t) |\nabla u(x)|^2\, |x_{n+1}|^a \,\mathrm{d}\mathcal{H}^n(x).\end{aligned}$$

An integration by parts then gives

$$D_u(x_0,r)=\frac{2}{r}\int_{\frac{r}{2}}^{r}dt\int_{B_t(x_0)}|\nabla u(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x. \tag{2.17}$$

Therefore, we can conclude the lower bound (2.15) by using the Poincaré inequality in [12, Lemma 2.13]

$$\frac{1}{t}\int_{\partial B_t(x_0)}|u(x)|^2\,|x_{n+1}|^a\,\mathrm{d}\mathcal{H}^n(x)\le C\int_{B_t(x_0)}|\nabla u(x)|^2\,|x_{n+1}|^a\,\mathrm{d}x. \qquad \square$$

We can then give the following compactness result which will be instrumental for the analysis we develop. To this aim it is mandatory to introduce the nodal set of u:

$$\mathcal{N}(u):=\Big\{(x',0)\in B'_R\,:\,u(x',0)=|\nabla_\tau u(x',0)|=\lim_{t\downarrow 0^+}t^a\partial_{n+1}u(x',t)=0\Big\}.$$

Notice that $\Gamma(u)\subseteq\mathcal{N}(u)$ by Corollary 2.4.

Corollary 2.10. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions to the thin obstacle problem (1.1) in B_1, with $\sup_k I_{u_k}(1)<+\infty$, $H_{u_k}(1)\le 1$ and $0\in\Gamma(u_k)$ for every $k\in\mathbb{N}$. Then, there exist a subsequence $(u_{k_j})_{j\in\mathbb{N}}\subset(u_k)_{k\in\mathbb{N}}$ and a solution u_0 to the thin obstacle problem in B_1 such that as $j\to\infty$*

$$u_{k_j}\to u_0\quad in\ H^1_{\mathrm{loc}}(B_1,|x_{n+1}|^a\,\mathcal{L}^{n+1}), \tag{2.18}$$

$$\nabla_\tau u_{k_j}\to\nabla_\tau u_0\quad in\ C^\alpha_{\mathrm{loc}}(B_1),\ \forall\,\alpha<s, \tag{2.19}$$

$$\mathrm{sign}(x_{n+1})\,|x_{n+1}|^a\partial_{x_{n+1}}u_{k_j}\to\mathrm{sign}(x_{n+1})\,|x_{n+1}|^a\partial_{x_{n+1}}u_0\ in\ C^\alpha_{\mathrm{loc}}(B_1),\ \forall\,\alpha<1-s, \tag{2.20}$$

$$u_{k_j}\to u_0\quad in\ C^\alpha_{\mathrm{loc}}(B_1),\ \forall\,\alpha<\min\{1,2s\}. \tag{2.21}$$

Moreover, if there is a sequence of points $x_{k_j}\in\Gamma(u_{k_j})$ such that $x_{k_j}\to x_0\in B_1$, then

$$x_0\in\mathcal{N}(u_0). \tag{2.22}$$

Proof. For every $t<1$, we have that

$$\int_{B_t}|\nabla u_k(x)|^2\,|x_{n+1}|^a\,dx\le\frac{D_{u_k}(1)}{2(1-t)}=\frac{I_{u_k}(1)\,H_{u_k}(1)}{2(1-t)}\le\frac{M}{2(1-t)},$$

where we have set for convenience $M:=\sup_k I_{u_k}(0,1)$. Moreover, from (2.14) we have that $\|u_k\|_{L^2(B_1,|x_{n+1}|^a\,\mathcal{L}^{n+1})}\le H_{u_k}(1)\le 1$. The sequence $(u_k)_{k\in\mathbb{N}}$ is equi-bounded in $H^1(B_t,|x_{n+1}|^a\,\mathcal{L}^{n+1})$ for every $t<1$. Therefore, (2.18) – (2.21) follow from Theorem 2.1 (cf. also [12, Lemma 4.4]). Moreover, since $x_{k_j}\in\Gamma(u_{k_j})\subseteq\mathcal{N}(u_{k_j})$, (2.22) follows from (2.19)–(2.21). $\square$

2.5. **Blow-up profiles.** An important consequence of the monotonicity of the frequency in Proposition 2.7 is the existence of blow-up profiles. For $u:B_R\to\mathbb{R}$ solution of (1.1) we introduce the rescalings

$$u_{x_0,r}(y):=\frac{r^{\frac{n+a}{2}}\,u(ry+x_0)}{H^{1/2}(x_0,r)}\quad\forall\,r\in(0,R-|x_0|),\ \forall\,y\in B_{\frac{R-|x_0|}{r}}. \tag{2.23}$$

Proposition 2.11. *Let u be a solution to the thin obstacle problem (1.1) in B_R. Then, for every $x_0\in\Gamma(u)$ and for every sequence of numbers $(r_j)_{j\in\mathbb{N}}\subset(0,1-|x_0|)$ with $r_j\downarrow 0$, there exists a subsequence $(r_{j_k})_{k\in\mathbb{N}}$ and function $u_0\in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1},|x_{n+1}|^a\mathcal{L}^{n+1})$ such that u_0 satisfies (1.1), u_0 is homogeneous of degree $I(x_0,0^+)$ and*

$$u_{x_0,r_{j_k}}\to u_0\quad in\ C^0_{\mathrm{loc}}(\mathbb{R}^{n+1})\cap H^1_{\mathrm{loc}}(\mathbb{R}^{n+1},|x_{n+1}|^a\mathcal{L}^{n+1}). \tag{2.24}$$

Proof. For every $\ell>0$, by Remark 2.6 we have $I_{u_{x_0,r_j}}(\ell)=I_u(x_0,r_j\,\ell)\le I_u(x_0,1-|x_0|)$. Therefore, from Corollary 2.8 we infer that there exists a constant $C=C(\ell)>0$ such that

$$H_{u_{x_0,r_j}}(\ell)\le C\,H_{u_{x_0,r_j}}(1)=C\qquad\forall\,r_j<\ell^{-1}(R-|x_0|).$$

We can then use Corollary 2.10 and a diagonal argument to infer the existence of a subsequence $(r_{j_k})_{k\in\mathbb{N}}$ and a solution u_0 such that (2.24) holds. We only need to show that u_0 is homogeneous. To this aim we notice that, by taking into account Lemma 2.9, we have for every $\ell > 0$

$$I_{u_0}(\ell) = \lim_{k\to\infty} I_{u_{x_0,r_{j_k}}}(\ell) = \lim_{k\to\infty} I_u(x_0, r_{j_k}\,\ell) = I_u(x_0, 0^+) \geq C_{2.9},$$

In particular, by Proposition 2.7 we conclude the homogeneity of u_0 of degree $I_u(x_0, 0^+)$. □

Corollary 2.12. *Let u be a solution to the thin obstacle problem (1.1) in B_R . Then,*

$$I_u(x_0, r) \geq 1 + s \quad \forall\, x_0 \in \Gamma(u),\ \forall\, r \in (0, R - |x_0|). \tag{2.25}$$

Proof. We consider the rescaling u_{x_0,r_j} and a blow-up limit u_0. By Proposition 2.11 we know that u_0 is homogeneous of degree $I_u(x_0, 0^+)$. Since solutions to (1.1) are $C^{1,s}(B_R)$ (cf. [12]), we easily conclude that $I_u(x_0, 0^+) \geq 1 + s$ and (2.25) follows by monotonicity. □

Remark 2.13. In general the limiting profile u_0 is not known to be unique. Uniqueness for $\mathcal{H}^{n-1}$-almost every free boundary point with infinitesimal homogeneity $2m$ and $2m - 1 + s$ will be established in Theorem 1.3, while uniqueness at every regular point follows from [12] (see also [20, 25] for an approach via the epiperimetric inequality) and at every singular point for $s = 1/2$ from [23].

Remark 2.14. It is more common in the literature to define the blow-up rescalings $\bar{u}_{x_0,r}$ as in (1.2). Nevertheless, by the same computations above, one can show that the height function $h_u(x_0, t) := \int_{\partial B_t(x_0)} u^2 \,\mathrm{d}\mathcal{H}^n$ satisfies the analogous monotonicity properties of Corollary 2.8 (see [12]) and moreover by (2.16) it is comparable to $H_u(x_0, t)$ (with a constant depending only on an upper bound of the frequency). In particular, this implies that the blow-ups with respect to these two different renormalizations only differ by a constant and all the results concerning them (*e.g.* the uniqueness) can be indifferently proven for either of the two definitions.

Due to our definition of the frequency, in the sequel we will always consider the rescalings defined in (2.23).

3. Main estimates on the frequency

In this section we prove the principal estimates on the frequency that we are going to exploit in the sequel.

Lemma 3.1. *For every $A > 0$ there exists $C_{3.1} = C_{3.1}(A) > 0$ such that, if u is a solution to the thin obstacle problem (1.1) in $B_{2r}(x_0)$, with $r > 0$, $x_0 \in \Gamma(u)$ and $I_u(x_0, 2r) \leq A$, then for every $x \in B'_{r/2}(x_0)$*

$$\frac{1}{C_{3.1}} \leq \frac{H_u(x_0, r)}{H_u(x, r)} \leq C_{3.1} \quad \text{and} \quad \frac{1}{C_{3.1}} \leq \frac{D_u(x_0, r)}{D_u(x, r)} \leq C_{3.1}, \tag{3.1}$$

$$\Big| I_u(x_0, r) - I_u(x, r) \Big| \leq C_{3.1}. \tag{3.2}$$

Proof. By rescaling it is enough to consider the case $x_0 = 0$, $r = 1$ and $H_u(0, 1) = 1$ (cf. Remark 2.6). In oder to prove (3.1), we argue by contradiction: assume there exists functions u_k and points $x_k \in B'_{1/2}$ contradicting the first inequality (3.1), *i.e.*

$$\lim_{k\to+\infty} H_{u_k}(x_k, 1) \in \{0, +\infty\}.$$

Note that, since $I_{u_k}(0, 2) \leq A$, it follows from (2.12) that $H_{u_k}(0, 2) \leq 2^{n+a+2A}$. In particular, we can apply Corollary 2.10 and (up to passing to a subsequence, not relabeled), there exist u_∞ and $x_\infty \in \bar{B}_{1/2}$ such that $u_k \to u_\infty$ in $H^1_{\mathrm{loc}}(B_2, |x_{n+1}|^a \mathcal{L}^{n+1})$ and $x_k \to x_\infty \in \bar{B}'_{1/2}$, with u_∞ solution to the thin obstacle problem in B_R for every $R < 2$. By the strong convergence of u_k to u_∞ we then deduce that $H_{u_\infty}(x_\infty, 1) \in \{0, \infty\} \cap \mathbb{R} = \{0\}$. Given that u_∞ is analytical in $B_2 \setminus \{x_{n+1} = 0\}$, by unique continuation we conclude that $u_\infty \equiv 0$ in B_2, against the assumption $H_{u_\infty}(0, 1) = \lim_k H_{u_k}(0, 1) = 1$.

The second inequality in (3.1) is proven by the same argument. Indeed, under the same assumption $H_u(0,1)=1$, considering that $0\in\Gamma(u)$, we have that $D_u(0,1)=I_u(0,1)\in[1+s,A]$. Therefore, given a sequence u_k contradicting the claim, we deduce the existence of a solution u_∞ such that $0=D_{u_\infty}(0,1)=\lim_k D_{u_\infty}(0,1)\in[1+s,A]$, which is impossible.

Finally, (3.2) follows straightforwardly from (3.1):

$$\Big|I_u(0,1)-I_u(x,1)\Big| = \Big|\frac{D_u(0,1)}{H_u(x,1)}\left(\frac{H_u(x,1)}{H_u(0,1)}-\frac{D_u(x,1)}{D_u(0,1)}\right)\Big| \overset{(3.1)}{\leq} C.$$

□

Lemma 3.2. *For every $A>0$ there exists $C_{3.2}=C_{3.2}(A)>0$ such that, if u is a solution to the thin obstacle problem (1.1) in $B_{2r_1}(x_0)\subset\mathbb{R}^{n+1}$ with $x_0\in\Gamma(u)$ and $I_u(x_0,2r_1)\leq A$, then for every $r_0\in({r_1}/8,r_1)$*

$$\int_{B_{r_1}(x_0)\setminus B_{r_0}(x_0)}\Big(\nabla u(z)\cdot(z-x_0)-I_u(x_0,r_0)\,u(z)\Big)^2\frac{|z_{n+1}|^a}{|z-x_0|}\,\mathrm{d}z \leq C_{3.2}H_u(x_0,2\,r_1)\left(I_u(x_0,2\,r_1)-I_u(x_0,r_0)\right). \tag{3.3}$$

Proof. By rescaling, it suffices to prove the lemma for $x_0=0$ and $r_1=1$. We start off with the following computation:

$$\begin{aligned}
2\int_{B_t\setminus B_{t/2}}&\Big(\nabla u(z)\cdot z-I_u(t)\,u(z)\Big)^2\frac{|z_{n+1}|^a}{|z|}\mathrm{d}z\\
&=\int -\phi'\big(\tfrac{|z|}{t}\big)\Big(\nabla u(z)\cdot z-I_u(t)\,u(z)\Big)^2\tfrac{|z_{n+1}|^a}{|z|}\mathrm{d}z\\
&=t^2E_u(t)-2\,t\,I_u(t)\,D_u(t)+I_u^2(t)H_u(t)\\
&=\frac{t^2}{H_u(t)}\Big[E_u(t)H_u(t)-D_u(t)^2\Big]\overset{(2.7)}{=}\frac{t}{2}\,I_u'(t)\,H_u(t).
\end{aligned}\tag{3.4}$$

We now use the following integral estimate (whose elementary proof is left to the readers)

$$\int_{B_1\setminus B_{r_0}}f(z)\mathrm{d}z\leq r_0^{-1}\int_{r_0}^2\int_{B_t\setminus B_{t/2}}f(z)\,\mathrm{d}z\,\mathrm{d}t\quad\forall\, f\geq 0,\ r_0\leq 1,\tag{3.5}$$

in order to deduce

$$\begin{aligned}
\int_{B_1\setminus B_{r_0}}&\Big(\nabla u(z)\cdot z-I_u(r_0)\,u(z)\Big)^2\frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\\
&\overset{(3.5)}{\leq} r_0^{-1}\int_{r_0}^2\int_{B_t\setminus B_{t/2}}\Big(\nabla u(z)\cdot z-I_u(r_0)\,u(z)\Big)^2\frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\,\mathrm{d}t\\
&\leq 2\,r_0^{-1}\int_{r_0}^2\int_{B_t\setminus B_{t/2}}\Big[\big(\nabla u(z)\cdot z-I_u(t)\,u(z)\big)^2+\big(I_u(t)-I_u(r_0)\big)^2u^2(z)\Big]\frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\,\mathrm{d}t\\
&\overset{(3.4)}{\leq} r_0^{-1}\int_{r_0}^2\frac{t\,H_u(t)}{2}\,I_u'(t)\,\mathrm{d}t+2\,r_0^{-1}\left(I_u(2)-I_u(r_0)\right)^2\int_{r_0}^2H_u(t)\,\mathrm{d}t.
\end{aligned}\tag{3.6}$$

Now recall that by (2.13) we have that $H_u(t)\leq H_u(2)$ for all $t\leq 2$. Hence, from (3.6) we get

$$\begin{aligned}
\int_{B_1\setminus B_{r_0}}&\Big(\nabla u(z)\cdot z-I_u(r_0)\,u(z)\Big)^2\frac{|z_{n+1}|^a}{|z|}\,\mathrm{d}z\\
&\leq r_0^{-1}\,H_u(2)\int_{r_0}^2I_u'(t)\,\mathrm{d}t+4r_0^{-1}\,H_u(2)\big(I_u(2)-I_u(r_0)\big)^2\leq C\,H_u(2)\left(I_u(2)-I_u(r_0)\right),
\end{aligned}$$

where we used that $r_0\geq\frac{1}{8}$ and and $I_u(2)\leq A$. □

3.1. **Oscillation estimate of the frequency.** We introduce the following notation for the radial variation of the frequency at a point $x \in \Gamma(u)$: given $0 < \rho < r$, we set

$$\Delta_\rho^r(x) := I_u(x, r) - I_u(x, \rho).$$

The following lemma shows how the spatial oscillation of the frequency in two nearby points at a given scale is in turn controlled by the radial variations at comparable scales. Here, we exploit for the thin obstacle problem an argument introduced in [16, Theorem 4.2] for multiple-valued functions.

Proposition 3.3. *For every $A > 0$ there exists $C_{3.3}(A) > 0$ such that, if $\rho > 0$, $R > 6$ and $u : B_{4R\rho}(x_0) \to \mathbb{R}$ is a solution to the thin obstacle problem (1.1) in $B_{4R\rho}$, with $x_0 \in \Gamma(u)$ and $I_u(x_0, 4R\rho) \le A$, then*

$$\big|I_u(x_1, R\rho) - I_u(x_2, R\rho)\big| \le C_{3.3} \left[\Big(\Delta^{2(R+2)\rho}_{(R-4)\rho/2}(x_1)\Big)^{1/2} + \Big(\Delta^{2(R+2)\rho}_{(R-4)\rho/2}(x_2)\Big)^{1/2}\right], \tag{3.7}$$

for every $x_1, x_2 \in B'_\rho$.

Proof. **1.** Without loss of generality, we show the proposition for $x_0 = 0$ and $\rho = 1$. The proof is based on estimating the tangential derivative of the frequency function $x \mapsto I_u(x, t)$ for a fixed radius $t > 0$. Thus, we start off noticing that the functions $x \mapsto H_u(x, t)$ and $x \mapsto D_u(x, t)$ are differentiable and, for every $e \in \mathbb{R}^{n+1}$ with $e \cdot e_{n+1} = 0$, we have that

$$\partial_e H_u(x, t) = -2 \int \phi'\big(\tfrac{|y|}{t}\big)\, u(y + x)\, \partial_e u(y + x)\, \tfrac{|y_{n+1}|^a}{|y|}\, \mathrm{d}y, \tag{3.8}$$

and

$$\begin{aligned} \partial_e D_u(x, t) &= 2 \int \phi\big(\tfrac{|y|}{t}\big) \nabla u(y + x) \cdot \nabla(\partial_e u)(y + x)\, |y_{n+1}|^a\, \mathrm{d}y \\ &= -2\, t^{-1} \int \phi'\big(\tfrac{|y|}{t}\big)\, \partial_e u(y + x)\, \nabla u(y + x) \cdot \tfrac{y}{|y|}\, |y_{n+1}|^a\, \mathrm{d}y, \end{aligned} \tag{3.9}$$

where the second equality follows from the divergence theorem applied to the vector field $V(y) := \phi\big(\frac{|y|}{t}\big)\, \partial_e u(y+x)\, |y_{n+1}|^a \nabla u(y+x)$ (note that $V \in C^\infty(B_t(x) \setminus B'_1)$, V has compact support and the divergence of V does not concentrate on B'_1 arguing as in proving formula (2.8) of Proposition 2.7). We consider next $e := x_2 - x_1$, and set

$$\mathcal{E}_i(z) := \nabla u(z) \cdot (z - x_i) - I_u(x_i, t)\, u(z) \quad \text{for } i = 1, 2,$$

$$\Delta I := I_u(x_1, t) - I_u(x_2, t) \quad \text{and} \quad \Delta\mathcal{E}(z) := \mathcal{E}_1(z) - \mathcal{E}_2(z).$$

Then, we have that $\partial_e u(z) = \Delta I \cdot u(z) + \Delta\mathcal{E}(z)$ and from (2.8), (3.8) – (3.9) we get also

$$\partial_e D_u(x, t) = 2\, \Delta I \cdot D_u(x, t) - 2\, t^{-1} \int \phi'\big(\tfrac{|y|}{t}\big)\, \Delta\mathcal{E}(y + x)\, \nabla u(y + x) \cdot \tfrac{y}{|y|}\, |y_{n+1}|^a\, \mathrm{d}y,$$

and

$$\partial_e H_u(x, t) = 2\Delta I \cdot H_u(x, t) - 2 \int \phi'\big(\tfrac{|y|}{t}\big)\, \Delta\mathcal{E}(y + x)\, u(y + x)\, \tfrac{|y_{n+1}|^a}{|y|}\, \mathrm{d}y.$$

In particular, by direct computation

$$\begin{aligned} \partial_e I(x, t) &= \frac{t}{H_u(x, t)^2} \big(H_u(x, t)\, \partial_e D_u(x, t) - D_u(x, t)\, \partial_e H_u(x, t)\big) \\ &= \frac{2}{H_u(x, t)} \int -\phi'\big(\tfrac{|y|}{t}\big)\, \Delta\mathcal{E}(y + x)\, \Big(\nabla u(y + x) \cdot y - I_u(x, t)\, u(y + x)\Big)\, \tfrac{|y_{n+1}|^a}{|y|}\, \mathrm{d}y. \end{aligned} \tag{3.10}$$

2. We use now (3.10) with $t = R$ and $x \in B_1'$. Note that, since $x \in B_1'$, by (2.1) – (2.3), (2.14) and (3.1) we infer that

$$\begin{aligned} M &:= \sup_{y\in B_R} |\nabla u(y+x)\cdot y - I_u(x,R)\,u(y+x)| \\ &\le \sup_{z\in B_{R+1}} \big(|\nabla u(z)\cdot z| + |\nabla_\tau u(z)|\big) + I_u(x,R)\|u\|_{C^0(B_{R+1})} \\ &\le C\,R^{-\frac{n+a}{2}}\,H_u^{1/2}\big(0,2R+2\big), \end{aligned}$$

for some constant $C = C(A) > 0$. Hence, we have that

$$\partial_e I_u(x,R) \le \frac{2M}{H_u(x,R)} \int -\phi'\big(\tfrac{|y|}{R}\big)\big(|\mathcal{E}_1(y+x)| + |\mathcal{E}_2(y+x)|\big)\tfrac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y. \tag{3.11}$$

In order to estimate the integral term in (3.11), we notice that

$$B_R(x)\setminus B_{R/2}(x) \subset B_{R+2}(x)\setminus B_{R/2-2}(x_i) \quad \forall\, x\in B_1',\ \text{for}\ \ i=1,2;$$

therefore

$$\begin{aligned} \int_{B_R\setminus B_{R/2}} |\mathcal{E}_i(y+x)|\,\frac{|y_{n+1}|^a}{|y|}\,\mathrm{d}y &\le \frac{2\,(R+2)}{R}\int_{B_{R+2}(x_i)\setminus B_{R/2-2}(x_i)} |\mathcal{E}_i(z)|\,\frac{|z_{n+1}|^a}{|z-x_i|}\,\mathrm{d}z \\ &\le C\,R^{\frac{n+a}{2}}\left(\int_{B_{R+2}(x_i)\setminus B_{R/2-2}(x_i)} \mathcal{E}_i^2(z)\,\frac{|z_{n+1}|^a}{|z-x_i|}\,\mathrm{d}z,\right)^{1/2}, \end{aligned} \tag{3.12}$$

where we used $R > 6$ and a direct computation to estimate

$$\int_{B_{R+2}(x_i)\setminus B_{R/2-2}(x_i)} \frac{|z_{n+1}|^a}{|z-x_i|}\,\mathrm{d}z \le C\,R^{n+a},$$

for a dimensional constant $C > 0$. We are in the position to apply Lemma 3.2:

$$\int_{B_{R+2}(x_i)\setminus B_{R/2-2}(x_i)} \mathcal{E}_i^2(z)\,\frac{|z_{n+1}|^a}{|z-x_i|}\,\mathrm{d}z \le C_{3.2}(A)\,H_u\big(x_i,2R+4\big)\,\Delta^{2(R+2)}_{R/2-2}(x_i). \tag{3.13}$$

Using (3.11) – (3.13), we get

$$\partial_e I_u(x,R) \le C\left(\Delta^{2(R+2)}_{R/2-2}(x_1)\right)^{1/2} + C\left(\Delta^{2(R+2)}_{R/2-2}(x_2)\right)^{1/2}, \tag{3.14}$$

having used (2.11) and (3.1) to infer that

$$H_u\big(0,2R+2\big)^{1/2}\,H_u\big(x,R\big)^{-1}H_u\big(x_i,2R+4\big)^{1/2} \le C(A)\,\frac{H_u(0,2R+4)}{H_u(0,R)} \le C(A).$$

In this respect, recall that $x, x_i \in B_1'$ and $R > 6$, so that we are in the position to apply Lemma 3.1. The conclusion now follows by integrating (3.14) along the segment $\{x_1 + r\,e : r\in[0,1]\}$. □

4. Mean-flatness and frequency function

4.1. **Mean-flatness.** We are going to use the following generalization of the Jones' β-numbers introduced in [27], also called *mean-flatness*, which have been already extensively used in the literature (cf., for example, [2, 6, 16, 31, 32] and the list of references therein). We adopt the standard notation $\mathrm{dist}(y,E) := \inf_{x\in E}|y-x|$ for the distance of a point y from a given subset $E \subset \mathbb{R}^{n+1}$.

Definition 4.1. Given a Radon measure μ in $\mathbb{R}^{n+1}$, $p\in[1,+\infty)$ and $k\in\{0,1,\dots,n\}$, for every $x_0\in\mathbb{R}^n$ and for every $r>0$, we set

$$\beta^{(k)}_{\mu,p}(x,r) := \inf_{\mathcal{L}}\left(r^{-k-p}\int_{B_r(x)} \mathrm{dist}(y,\mathcal{L})^p\,\mathrm{d}\mu(y)\right)^{\frac{1}{p}}, \tag{4.1}$$

where the infimum is taken among all affine k-dimensional planes $\mathcal{L}\subset\mathbb{R}^{n+1}$.

In the case $p = 2$ we have the following elementary characterization. Let $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$ be such that $\mu(B_r(x_0)) > 0$, and let us denote by $\bar{x}_{x_0,r}$ the barycenter of μ in $B_r(x_0)$, *i.e.*

$$\bar{x}_{x_0,r} := \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} x \, \mathrm{d}\mu(x).$$

Let moreover $\mathbf{B}_{x_0} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the symmetric positive semi-definite bilinear form given by

$$\mathbf{B}_{x_0}(v, w) := \int_{B_r(x_0)} \big((x - \bar{x}_{x_0,r}) \cdot v\big) \, \big((x - \bar{x}_{x_0,r}) \cdot w\big) \, \mathrm{d}\mu(x) \quad \forall \, v, \, w \in \mathbb{R}^{n+1}.$$

By standard linear algebra there exists an orthonormal basis of vectors in $\mathbb{R}^{n+1}$ which diagonalizes the bilinear form $\mathbf{B}_{x_0}$: namely, there exist $\{v_1, \ldots, v_{n+1}\} \subset \mathbb{R}^{n+1}$ (in general not uniquely determined) such that

(i) $\{v_1, \ldots, v_{n+1}\}$ is a Euclidean orthonormal basis, *i.e.* $v_i \cdot v_j = \delta_{ij}$;
(ii) $\mathbf{B}_{x_0}(v_i, v_i) = \lambda_i$, for some $0 \le \lambda_{n+1} \le \lambda_n \le \cdots \le \lambda_1$;
(iii) $\mathbf{B}_{x_0}(v_i, v_j) = 0$ for $i \neq j$.

The characterization is then the following: the infimum in the definition of $\beta^{(k)}_{\mu,2}$ is reached by all the affine planes $\mathcal{L} = \bar{x}_{x_0,r} + \mathrm{Span}\{v_1, \ldots, v_k\}$ and

$$\int_{B_r(x_0)} \big((x - \bar{x}_{x_0,r}) \cdot v_i\big) \, x \, \mathrm{d}\mu(x) = \lambda_i \, v_i \quad \forall \, i = 1, \ldots, n+1, \tag{4.2}$$

$$\beta^{(k)}_{\mu,2}(x_0, r) = \Big(r^{-k-2} \sum_{l=k+1}^{n+1} \lambda_l \Big)^{\frac{1}{2}}. \tag{4.3}$$

In the ensuing sections we are going to consider only the case $k = n-1$ and $p = 2$: in order to simplify the notation we will always write β_μ for $\beta^{(n-1)}_{\mu,2}$.

4.2. **Control of the mean-flatness via the frequency.** The main link between Jones' β-numbers and the geometric properties of the free boundary is given by the following observation: the mean-flatness of an arbitrary measure μ supported on $\Gamma(u)$ is controlled by the integration with respect to μ of suitable radial oscillations of the frequency. This follows closely the approach by Naber and Valtorta [31, Theorem 7.1] for harmonic maps and minimal surfaces. Because of the intrinsic renormalization of the frequency function here we need to use the estimate in Proposition 3.3 as done in [16] for multiple-valued functions.

Proposition 4.2. *For every $A > 0$ and $R > 5$, there exists a constant $C_{4.2}(A, R) > 0$ with this property. Let $r > 0$, $u \in H^1(B_{(4R+10)r}(x_0), |x_{n+1}|^a \mathcal{L}^{n+1})$ be a solution to the thin obstacle problem (1.1) in $B_{(4R+10)r}(x_0)$, with $x_0 \in \Gamma(u)$ and $I(x_0, (4R+10)r) \le A$, and let μ be a finite Borel measure with* $\mathrm{spt}\,(\mu) \subseteq \Gamma(u)$*; then*

$$\beta^2_\mu(p, r) \le \frac{C_{4.2}}{r^{n-1}} \int_{B_r(p)} \Delta^{(2R+4)\,r}_{(R-5)\,r/2}(x) \, \mathrm{d}\mu(x) \quad \forall \, p \in \Gamma(u) \cap B_r. \tag{4.4}$$

Proof. **1.** We can assume without loss of generality that $x_0 = 0$ and that $p \in \Gamma(u) \cap B_r$ is such that $\mu(B_r(p)) > 0$ (otherwise, there is nothing to prove). Let $\bar{x} = \bar{x}_{p,r}$ be the barycenter of μ in $B_r(p)$ and let $\{v_1, \ldots, v_{n+1}\}$ be any diagonalizing basis for the bilinear form $\mathbf{B}_p$ introduced in § 4.1, with corresponding eigenvalues $0 \le \lambda_{n+1} \le \lambda_n \le \cdots \le \lambda_1$. Note that, since by assumption $\mathrm{spt}\,(\mu) \subset \Gamma(u) \subset \mathbb{R}^n \times \{0\}$, we can assume without loss of generality that $v_{n+1} = e_{n+1}$, $\lambda_{n+1} = 0$ and hence $\beta_\mu(p, r) = (r^{-n-1}\lambda_n)^{1/2}$ by (4.3). Therefore, without loss of generality we may also assume that $\lambda_n > 0$.

From (4.2) and the definition of barycenter we deduce that, for every $i \in \{1, \ldots, n\}$, for every $z \in B_{(2R+5)r}$ and for every constant $\alpha \in \mathbb{R}$, we have

$$-\lambda_i \, v_i \cdot \nabla u(z) = \int_{B_r(p)} \big((x - \bar{x}) \cdot v_i\big) \, \big((z - x) \cdot \nabla u(z) - \alpha \, u(z)\big) \mathrm{d}\mu(x).$$

For the rest of the proof we set

$$\alpha := \frac{1}{\mu(B_r)} \int_{B_r(p)} I_u(x, (R-1)\, r) \mathrm{d}\mu(x).$$

Using Hölder inequality we deduce that

$$\begin{aligned}
\lambda_i^2 |v_i \cdot \nabla u(z)|^2 &\le \int_{B_r(p)} \big((x-\bar{x}) \cdot v_i\big)^2 \mathrm{d}\mu(x) \int_{B_r(p)} \big((z-x)\cdot \nabla u(z) - \alpha\, u(z)\big)^2 \mathrm{d}\mu(x) \\
&\overset{\S 4.1\,\text{(ii)}}{=} \lambda_i \int_{B_r(p)} \big((z-x)\cdot \nabla u(z) - \alpha\, u(z)\big)^2 \mathrm{d}\mu(x). \qquad (4.5)
\end{aligned}$$

Denoting with $\nabla_\tau u = (\partial_1 u, \ldots, \partial_n u)$ the tangential gradient, and recalling that

$$r^{n+1} \beta_\mu^2(p,r) = \lambda_n \quad \text{and} \quad 0 = \lambda_{n+1} < \lambda_n \le \ldots \le \lambda_1,$$

by integrating over $B_{(R+1)r}(p) \setminus B_{Rr}(p)$ the previous inequality with respect to the measure $|z_{n+1}|^a \mathcal{L}^{n+1}$ we get

$$\begin{aligned}
r^{n+1}\beta_\mu^2(p,r) &\int_{B_{(R+1)r}(p)\setminus B_{Rr}(p)} |\nabla_\tau u(z)|^2 |z_{n+1}|^a \mathrm{d}z = \lambda_n \int_{B_{(R+1)r}(p)\setminus B_{Rr}(p)} |\nabla_\tau u(z)|^2 |z_{n+1}|^a \mathrm{d}z \\
&\le \int_{B_{(R+1)r}(p)\setminus B_{Rr}(p)} \sum_{i=1}^n \lambda_i |v_i \cdot \nabla u(z)|^2 |z_{n+1}|^a \,\mathrm{d}z \\
&\overset{(4.5)}{\le} n \int_{B_{(R+1)r}(p)\setminus B_{Rr}(p)} \int_{B_r(p)} \big((z-x)\cdot \nabla u(z) - \alpha\, u(z)\big)^2 \mathrm{d}\mu(x)\, |z_{n+1}|^a \mathrm{d}z \\
&\le n \int_{B_r(p)} \int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} \big((z-x)\cdot \nabla u(z) - \alpha\, u(z)\big)^2 |z_{n+1}|^a \mathrm{d}z \, \mathrm{d}\mu(x). \qquad (4.6)
\end{aligned}$$

Next we estimate the two sides of (4.6).

2. For what concerns the l.h.s. of (4.6), we claim the following: there exists a constant $C = C(A,R) > 0$ such that

$$D_u(p, (R+2)r) \le C \int_{B_{(R+1)r}(p)\setminus B_{Rr}(p)} |\nabla_\tau u(z)|^2 |z_{n+1}|^a \mathrm{d}z. \qquad (4.7)$$

We argue by contradiction. If the claim were not true, after a suitable rescaling replacing u with $u_{p,r}$, we could find a sequence of solutions u_k to the thin obstacle problem in B_{2R+4} with $0 \in \Gamma(u_k)$, such that $I_{u_k}(R+3) \le A' := A + C_{3.1}(A)$, $H_{u_k}(R+2) = 1$ and

$$\int_{B_{R+1}\setminus B_R} |\nabla_\tau u_k(z)|^2 |z_{n+1}|^a \mathrm{d}z \le \frac{D_{u_k}(R+2)}{k},$$

(recall that $B_{(2R+4)r}(p) \subset B_{(4R+10)r}$ and by Lemma 3.1 we have $I_u(p, (R+3)r) \le A'$). By Corollary 2.8 we have that $H_{u_k}(R+3) \le \big({}^{(R+3)}/_{(R+2)}\big)^{n+a+2A'}$ and hence by Corollary 2.10, (up to subsequences, not relabeled) u_k converge in $H^1(B_{R+2}, |x_{n+1}|^a \mathcal{L}^{n+1})$ to a solution u_0 to the thin obstacle problem in B_{R+2} with $H_{u_0}(R+2) = 1$ and

$$\int_{B_{R+1}\setminus B_R} |\nabla_\tau u_0(z)|^2 |z_{n+1}|^a \mathrm{d}z = 0.$$

We deduce from the latter equality that u_0 depends only on the variable x_{n+1} (recall that u_0 is analytic in B_1^+). In particular, $u_0(x) = -c\,|x_{n+1}|^{2s}$ for some $c > 0$, and

$$I_{u_0}(t) = 2s < 1 + s \le I_{u_k}(t) \quad \forall\, t \in (0, R+2),\ \forall\, k \in \mathbb{N},$$

where we used Lemma 2.9. This contradicts $\lim_k I_{u_k}(t) = I_{u_0}(t)$ and concludes the proof of (4.7).

3. Now we estimate the r.h.s. of (4.6) from above. By the triangular inequality we have that

$$\text{r.h.s. of (4.6)} \leq$$
$$\leq 2n \int_{B_r(p)} \int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} \big((z-x)\cdot\nabla u(z) - I_u(x,(R-1)r)\,u(z)\big)^2 |z_{n+1}|^a \,\mathrm{d}z\,\mathrm{d}\mu(x)$$
$$+ 2n \int_{B_r(p)} \int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} \Big(I_u(x,(R-1)r) - \alpha\Big)^2 u^2(z)|z_{n+1}|^a \,\mathrm{d}z\,\mathrm{d}\mu(x).$$

For every $x \in \mathrm{spt}\,(\mu)\cap B_r(p)$, (3.2) in Lemma 3.1 yields

$$I_u(x,(R+2)r) \leq I_u(0,(R+2)r) + C_{3.1}(A) \leq A + C_{3.1}(A) = A',$$

since $B_r(p) \subseteq B_{2r}$ and u is defined on $B_{(2R+4)r}(p) \subset B_{(4R+10)r}$. By using Lemma 3.2, we can estimate the first integral above as follows:

$$\int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} \big((z-x)\cdot\nabla u(z) - I_u(x,(R-1)r)\,u(z)\big)^2 |z_{n+1}|^a \,\mathrm{d}z$$
$$\leq (R+2)\,r \int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} \big((z-x)\cdot\nabla u(z) - I_u(x,(R-1)r)\,u(z)\big)^2 \frac{|z_{n+1}|^a}{|z-x|}\,\mathrm{d}z$$
$$\overset{(3.3)}{\leq} C_{3.2}(A')\,(R+2)\,r\,H\big(x,(2R+4)r\big)\,\Delta^{(2R+4)r}_{(R-1)r}(x)$$
$$\leq C(A)\,(R+2)\,r\,H\big(p,(2R+4)r\big)\,\Delta^{(2R+4)r}_{(R-1)r}(x), \tag{4.8}$$

in the last inequality we have used Lemma 3.1 (because $|p-x| < r$ and u is defined in $B_{(4R+8)r}(p) \subset B_{(4R+10)r}$). On the other hand, using Jensen's inequality and Proposition 3.3 (recall that $\mathrm{spt}\,(\mu) \subseteq \Gamma(u)$), we deduce that

$$\int_{B_r(p)} \Big(I_u(x,(R-1)r) - \alpha\Big)^2 \mathrm{d}\mu(x)$$
$$\leq \frac{1}{\mu(B_r(p))} \int_{B_r(p)} \int_{B_r(p)} \Big(I_u\big(x,(R-1)r\big) - I_u\big(y,(R-1)r\big)\Big)^2 \mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$
$$\overset{(3.7)}{\leq} \frac{2\,C^2_{3.3}(A')}{\mu(B_r(p))} \int_{B_r(p)} \int_{B_r(p)} \Big(\Delta^{2(R+1)r}_{(R-5)^r/_2}(x) + \Delta^{2(R+1)r}_{(R-5)^r/_2}(y)\Big)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$
$$\leq C \int_{B_r(p)} \Delta^{2(R+1)r}_{(R-5)^r/_2}(x)\,\mathrm{d}\mu(x). \tag{4.9}$$

Finally, note that

$$\int_{B_{(R+2)r}(x)\setminus B_{(R-1)r}(x)} u^2(z)|z_{n+1}|^a\,\mathrm{d}z \overset{(2.14)}{\leq} (R+2)\,r\,H_u\big(x,(R+2)r\big)$$
$$\overset{(3.1)}{\leq} C\,(R+2)\,r\,H_u\big(p,(R+2)r\big), \tag{4.10}$$

where once again in the last inequality we have used Lemma 3.1.

4. We can now collect the estimates (4.7) – (4.10) to get

$$r^{n+1}\beta^2_\mu(p,r)D_u(p,(R+2)r) \leq C(A,R)\,r\,H_u\big(p,(2R+4)r\big) \int_{B_r(p)} \Delta^{2(R+1)r}_{(R-5)^r/_2}(x)\,\mathrm{d}\mu(x)$$
$$\overset{(2.11)}{\leq} C(A,R)\,r\,H_u\big(p,(R+2)r\big) \int_{B_r(p)} \Delta^{2(R+1)r}_{(R-5)^r/_2}(x)\,\mathrm{d}\mu(x).$$

From $I_u(p,(R+2)r) \geq 1+s$ (cf. Corollary 2.12), one can then infer (4.4). □

5. Homogeneous and almost homogeneous solutions

For the proof of the main theorems, we need to discuss some results concerning homogeneous and almost homogeneous solutions to the thin obstacle problem (1.1).

5.1. **Spines of homogeneous solutions.** We denote by $\mathcal{H}_\lambda$ the space of all (non-trivial) λ-homogeneous solutions to the thin obstacle problem (1.1),

$$\mathcal{H}_\lambda := \Big\{u \in H^1_{\mathrm{loc}}\big(\mathbb{R}^{n+1}, |x_{n+1}|^a\,\mathcal{L}^{n+1}\big) \setminus \{0\} : u(x) = |x|^\lambda\, u\big(x/|x|\big),\ u|_{B_1} \text{ solves } (1.1)\Big\},$$

and set $\mathcal{H} := \bigcup_{\lambda \geq 1+s} \mathcal{H}_\lambda$. The restriction $\lambda \geq 1+s$ is imposed in view of Corollary 2.12. Recall next the definition of spine of homogeneous solutions (see, *e.g.*, [19]).

Definition 5.1. Let $u \in \mathcal{H}$ be a homogeneous solution. The *spine* $S(u)$ is the maximal subspace of invariance of u:

$$S(u) := \Big\{y \in \mathbb{R}^n \times \{0\} \ :\ u(x+y) = u(x) \quad \forall\, x \in \mathbb{R}^{n+1}\Big\}.$$

Simple characterizations of the spine are provided in the next result.

Lemma 5.2. *Let $u \in \mathcal{H}$ be given. The following are equivalent conditions:*

(i) $x \in S(u)$,
(ii) *u is homogeneous with respect to x, i.e.* $I_u(x,r) = I_u(x,0^+)$ *for all* $r > 0$,
(iii) $I_u(x,0^+) = I_u(0,0^+)$.

Proof. The very definition of spine yields straightforwardly that (i) implies (ii) and (iii). To see that (ii) implies (iii), we consider the functions u_{0,r_k} as defined in (2.23), for a sequence of radii $r_k \uparrow +\infty$ such that u_{0,r_k} converge to some u_∞ in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, |x_{n+1}|^a\mathcal{L}^{n+1})$. Then, by Remark 2.6 we infer that

$$\begin{aligned}\big|I_u(x,0^+) - I_u(0,0^+)\big| &\overset{\text{(ii)}}{=} \lim_{k\to+\infty} |I_u(x,r_k) - I_u(0,r_k)| = \lim_{k\to+\infty} \Big|I_{u_{0,r_k}}(x/r_k, 1) - I_{u_{0,r_k}}(0,1)\Big| \\ &= |I_{u_\infty}(0,1) - I_{u_\infty}(0,1)| = 0.\end{aligned}$$

Similarly, (iii) implies (ii): let $r_k \uparrow +\infty$ be a sequence as above, then using $u \in \mathcal{H}$ we get

$$\begin{aligned}\lim_{k\to+\infty} \big|I_u(x,r_k) - I_u(x,0^+)\big| &\leq \lim_{k\to+\infty} |I_u(x,r_k) - I_u(0,r_k)| + \big|I_u(x,0^+) - I_u(0,0^+)\big| \\ &\overset{\text{(iii)}}{=} \lim_{k\to+\infty} |I_u(x,r_k) - I_u(0,r_k)| = 0.\end{aligned}$$

In particular, taking into account the monotonicity of the frequency, we infer that $I_u(x,r) = I_u(x,0^+)$ for every $r > 0$, *i.e.* (ii). Finally, we are left to show that (ii) and (iii) imply (i). By (ii) and (iii) we have that

$$u(y+x) = t^\lambda\, u\left(\frac{y}{t} + x\right) \quad \forall\, y \in \mathbb{R}^n,\ \forall\, t > 0,$$

with $\lambda = I_u(0,0^+)$. Hence, for every $y \in \mathbb{R}^n \times \{0\}$ we have

$$u(y) = u(x + y - x) = 2^\lambda\, u\big(x + (y-x)/2\big) = 2^\lambda\, u\big((y+x)/2\big) = u(y+x). \qquad \square$$

5.2. **Classification of solutions with maximal dimension of the spine.** There are no homogeneous functions $u \in \mathcal{H}$ with spine of dimension n, because the only non-trivial solutions of (1.1) even with respect to x_{n+1} and depending only on the variable x_{n+1} are of the form $c\,|x_{n+1}|^{2s}$ with $c < 0$. The latter functions have homogeneity $2s < 1+s$, that is not allowed for functions in $\mathcal{H}$. Therefore, the maximal dimension of the spine of a function in $\mathcal{H}$ is at most $n-1$. We say that $u \in \mathcal{H}^{\mathrm{top}}$ if $u \in \mathcal{H}$ and $\dim S(u) = n-1$, and we set $\mathcal{H}^{\mathrm{low}} := \mathcal{H} \setminus \mathcal{H}^{\mathrm{top}}$ otherwise. All functions in $\mathcal{H}^{\mathrm{top}}$ are classified in the following list.

Lemma 5.3. *$u \in \mathcal{H}^{\mathrm{top}}$ if and only if there exists a uniquely determined λ-homogeneous function $h_\lambda : \mathbb{R}^2 \to \mathbb{R}$, with $\lambda \in \{2m, 2m-1+s, 2m+2s\}_{m \in \mathbb{N}\setminus\{0\}}$, such that*

$$u(x) = c\, h_\lambda(x \cdot e, x_{n+1}) \quad \text{and} \quad H_{h_\lambda(x_1,x_{n+1})}(0,1) = 1,$$

for some $c > 0$ and $e \in \mathbb{R}^n \times \{0\}$ with $|e| = 1$. In particular, if $u \in \mathcal{H}^{\mathrm{top}}$ then $\mathcal{N}(u) = S(u)$, and more precisely: if $u(x) = c\, h_\lambda(x \cdot e, x_{n+1})$, then

(I) *if $\lambda = 2m$:* $\Lambda(u) = \Gamma(u) = \mathcal{N}(u) = S(u) = \big\{x \cdot e = x_{n+1} = 0\big\}$;

(II) *if* $\lambda = 2m-1+s$*:* $\Lambda(u) = \{x \cdot e \leq 0,\, x_{n+1} = 0\}$ *and* $\Gamma(u) = \mathcal{N}(u) = S(u) = \{x \cdot e = x_{n+1} = 0\}$*;*

(III) *if* $\lambda = 2m+2s$*:* $\Lambda(u) = \{x_{n+1} = 0\}$, $\Gamma(u) = \emptyset$ *and* $\mathcal{N}(u) = S(u) = \{x \cdot e = x_{n+1} = 0\}$.

The proof of Lemma 5.3 is a consequence of the full characterization of the homogeneous solutions $v : \mathbb{R}^2 \to \mathbb{R}$ to the thin obstacle problem. Introducing polar co-ordinates $v(\rho\cos\theta, \rho\sin\theta) = \rho^\lambda\, y(\theta)$ with $y : [0,\pi] \to \mathbb{R}$, the system (1.1) can be written in the form:

$$y''(\theta) + a\mathrm{cotg}\,\theta\, y'(\theta) + \lambda\,(\lambda + a)\, y(\theta) = 0, \tag{5.1}$$

with the following four possible boundary conditions:

$$y(0) > 0, \quad y(\pi) > 0 \quad \text{and} \quad \lim_{\theta\downarrow 0^+} (\sin\theta)^{1-2s} y'(\theta) = \lim_{\theta\downarrow \pi^-} (\sin\theta)^{1-2s} y'(\theta) = 0, \tag{5.2}$$

$$y(0) = 0 < y(\pi), \quad \lim_{\theta\downarrow 0} (\sin\theta)^{1-2s} y'(\theta) \leq 0 \quad \text{and} \quad \lim_{\theta\uparrow \pi} (\sin\theta)^{1-2s} y'(\theta) = 0, \tag{5.3}$$

$$y(\pi) = 0 < y(0), \quad \lim_{\theta\downarrow 0} (\sin\theta)^{1-2s} y'(\theta) = 0 \quad \text{and} \quad \lim_{\theta\uparrow \pi} (\sin\theta)^{1-2s} y'(\theta) \geq 0, \tag{5.4}$$

$$y(0) = y(\pi) = 0, \quad \lim_{\theta\downarrow 0} (\sin\theta)^{1-2s} y'(\theta) \leq 0 \quad \text{and} \quad \lim_{\theta\uparrow \pi} (\sin\theta)^{1-2s} y'(\theta) \geq 0. \tag{5.5}$$

The four cases (5.2) – (5.5) determine the corresponding exponents λ and the solutions h_λ as in (I), (II), (III) of the lemma. The proof in general requires the use of the associated Legendre functions and is postponed to the appendix where we establish also other complementary results that are mandatory for the analysis in Section 8 (cf. Proposition A.1). Here we give the details for the simplest case of the Signorini problem $s = 1/2$, *i.e.* $a = 0$.

Proof of Lemma 5.3 for $s = 1/2$. If $a = 0$, the general integral of (5.1) is

$$y(\theta) = A_1\, \cos(\lambda\theta) + A_2\, \sin(\lambda\theta),$$

with $A_1, A_2 \in \mathbb{R}$. We can then discuss the four possible cases.

(I) For (5.2) we have that $y'(0) = 0$ implies $A_2 = 0$ and $A_1 \neq 0$, and $y'(\pi) = 0$ gives $\lambda \in \mathbb{N}$. Considering that $y(0) > 0$ we find $A_1 > 0$, and by $y(\pi) > 0$ one gets $\lambda = 2m$ with $m \in \mathbb{N} \setminus \{0\}$.

(II) For (5.3), we have that $y(0) = 0$ gives $A_1 = 0$ and $A_2 \neq 0$. In turn $y'(0) \leq 0$ implies $A_2 < 0$. Thus, $y'(\pi) = 0$ yields $\cos(\lambda\pi) = 0$, *i.e.* $\lambda = m + 1/2$ for $m \in \mathbb{N} \setminus \{0\}$, and finally m is odd since $y(\pi) > 0$. One proceeds analogously in case (5.4).

(III) Finally, for (5.5), we have that $y(0) = y(\pi) = 0$ implies $A_1 = 0$, $A_2 \neq 0$ and $\lambda \in \mathbb{N}$. Considering that $y'(0) \leq 0 \leq y'(\pi)$ we conclude that $A_2 < 0$ and λ odd.

In all the cases the nonzero coefficient A_i is chosen suitably in order to satisfy the normalization condition $H_{h_\lambda}(1) = 1$. The statements concerning $\Gamma(u)$, $\Lambda(u)$, $\mathcal{N}(u)$ and $S(u)$ are now direct consequences of the explicit formulas for the solutions. □

For the lowest frequency $1+s$, actually all homogeneous solutions have maximal spine as proved by Caffarelli, Salsa and Silvestre in [12], this result can be equivalently stated by the inclusion

$$\mathcal{H}_{1+s} \subseteq \mathcal{H}^{top}. \tag{5.6}$$

5.3. **Almost homogeneous solutions.** We next introduce the notion of almost homogeneous solutions.

Definition 5.4. Let $\eta > 0$ and $R > 1$ be given constants. A solution $u : B_R \to \mathbb{R}$ to thin obstacle problem (1.1) is called η*-almost homogeneous* if $0 \in \Gamma(u)$ and

$$I_u(1) - I_u(1/2) \leq \eta.$$

The following lemma justifies this terminology.

Lemma 5.5. *For every* $\delta, A > 0$ *and* $R > 1$ *there exists* $\eta > 0$ *with the following property: let* u *be a* η*-almost homogeneous solution with* $I_u(R) \leq A$ *and* $H_u(R) = 1$*; then, there exists a homogeneous solution* $w \in \mathcal{H}$ *such that*

$$\|u - w\|_{H^1(B_{R-1}, |x_{n+1}|^a \mathcal{L}^{n+1})} \leq \delta. \tag{5.7}$$

Proof. We argue by contradiction: assume there exist δ, A, R as in the statement and a sequence $(u_k)_{k\in\mathbb{N}}$ of k^{-1}-homogeneous solutions in B_R with $I_{u_k}(R) \le A$ such that

$$H_{u_k}(R) = 1 \quad \text{and} \quad \inf_{w\in\mathcal{H}} \|u_k - w\|_{H^1(B_{R-1},|x_{n+1}|^a\,\mathcal{L}^{n+1})} > \delta > 0. \tag{5.8}$$

We can then apply Corollary 2.10 and find a subsequence (not relabeled) and a solution u_0 to the obstacle problem in B_R such that $u_k \to u_0$ in $H^1(B_{R-1}, |x_{n+1}|^a\,\mathcal{L}^{n+1})$. Note that

$$H_{u_0}(R-1) = \lim_{k\to+\infty} H_{u_k}(R-1) \overset{(2.12)}{\ge} \Big(\frac{R-1}{R}\Big)^{n+a+2A} \lim_{k\to+\infty} H_{u_k}(R) = \Big(\frac{R-1}{R}\Big)^{n+a+2A}.$$

In particular, u_0 is non-trivial and

$$I_{u_0}(1) - I_{u_0}(1/2) = \lim_{k\to+\infty}\Big(I_{u_k}(1) - I_{u_k}(1/2)\Big) = 0.$$

By Proposition 2.7 we infer that u_0 is homogeneous of degree $I_{u_0}(1) = \lim_{k\uparrow+\infty} I_{u_k}(1) \ge 1+s$, because $0\in\Gamma(u_k)$. Therefore, $u_0\in\mathcal{H}$ and this contradicts (5.8). ☐

Next we show a rigidity result which will be used crucially in the estimate of the Hausdorff measure of the free boundary.

Proposition 5.6. *For every $\tau, A > 0$ there exists $\eta_{5.6}(\tau, A) > 0$ with this property. Let $u : B_4 \to \mathbb{R}$ be a $\eta_{5.6}$-almost homogeneous solution to the thin obstacle problem with $I_u(0,4)\le A$. Then, the following dichotomy holds:*

(i) *either for every point $x\in\Gamma(u)\cap B_2$ we have*

$$|I_u(x,1) - I_u(0,1)| \le \tau, \tag{5.9}$$

(ii) *or there exists a linear subspace $V\subset\mathbb{R}^n\times\{0\}$ of dimension $n-2$ such that*

$$y\in\Gamma(u)\cap B_2,\ I_u(y,1) - I_u\big(y,1/2\big) \le \eta_{5.6} \quad\Longrightarrow\quad y\in\mathcal{T}_\tau(V), \tag{5.10}$$

recall the notation $\mathcal{T}_\tau(V) := \{x\in\mathbb{R}^{n+1} : \mathrm{dist}(x,V) < \tau\}$.

Proof. The proof is by contradiction. We assume that there exist τ, A as in the statement and a sequence $(u_k)_{k\in\mathbb{N}}$ of k^{-1}-almost homogeneous solutions in B_4 with $I_{u_k}(4)\le A$ contradicting the statement, *i.e.* both (i) and (ii) simultaneously fail: namely, there exists $x_k\in\Gamma(u_k)\cap B_2$ such that

$$|I_{u_k}(x_k,1) - I_{u_k}(0,1)| > \tau, \tag{5.11}$$

and for every linear subspace $V\in\mathbb{R}^n\times\{0\}$ of dimension $n-2$ there exists $y_k\in\Gamma(u_k)\cap B_2$ (a priori depending on V) such that

$$I_{u_k}(y_k,1) - I_{u_k}\big(y_k,1/2\big) \le k^{-1} \quad\text{and}\quad y_k\notin\mathcal{T}_\tau(V). \tag{5.12}$$

By eventually rescaling the functions of the sequence, we can assume without loss of generality that $H_{u_k}(0,4) = 1$. In particular, it follows from Lemma 5.5 that

$$\lim_{k\to+\infty}\inf_{w\in\mathcal{H}} \|u_k - w\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})} = 0. \tag{5.13}$$

Up to passing to a subsequence (not relabeled) we distinguish to cases:

(a) either there exists $w_k\in\mathcal{H}^{\mathrm{top}}$ such that $\lim_k \|u_k - w_k\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})} = 0$;

(b) or there exists $\delta > 0$ such that

$$\inf_{w\in\mathcal{H}^{\mathrm{top}}} \|u_k - w\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})} \ge \delta > 0 \quad \forall\, k\in\mathbb{N}. \tag{5.14}$$

In case (a) we show that (5.11) cannot hold. Indeed, by Corollary 2.10 there exist a homogeneous function $w_0\in\mathcal{H}^{\mathrm{top}}$ (note that $\mathcal{H}^{\mathrm{top}}$ is closed under locally strong H^1 convergence), a point $x_0\in\bar{B}_2$ and a subsequence (not relabeled) such that

$$\lim_{k\to+\infty}\Big(\|u_k - w_0\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})} + \|w_k - w_0\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})}\Big) = 0$$

$$x_k \to x_0 \in \mathcal{N}(w_0)\cap\bar{B}_2.$$

In particular,

$$|I_{w_0}(x_0,1) - I_{w_0}(0,1)| = \lim_{k\to+\infty} |I_{u_k}(x_k,1) - I_{u_k}(0,1)| \geq \tau.$$

Note that $w_0 \not\equiv 0$, because $H_{w_0}(0,3) = \lim_k H_{u_k}(0,3) \geq (3/4)^{n+a+2A}$ thanks to (2.12) in Corollary 2.8. This implies that $I_{w_0}(x_0,1) = I_{w_0}(0,1)$, since $x_0 \in \mathcal{N}(w_0) = S(w_0)$ being $w_0 \in \mathcal{H}^{\mathrm{top}}$ and Lemma 5.3, which gives the desired contradiction.

In case (b), by combining (5.13) and (5.14), and by the compactness in Corollary 2.10 (up to passing to a subsequence, not relabeled) there exists $v_0 \in \mathcal{H}$ such that

$$\lim_{k\to+\infty} \|u_k - v_0\|_{H^1(B_3,|x_{n+1}|^a\,\mathcal{L}^{n+1})} = 0.$$

Moreover, from (5.14) we deduce that $v_0 \in \mathcal{H}^{\mathrm{low}}$ (note that $v_0 \not\equiv 0$ because we have that $H_{v_0}(0,3) = \lim_k H_{u_k}(0,3) \geq (3/4)^{n+a+2A}$ by Corollary 2.8). We now show that we have a contradiction to (5.12) with V any $(n-2)$-dimensional subspace containing $S(v_0)$. Indeed, let y_k be as in (5.12) for such a choice of V; by compactness, up to passing to a subsequence (not relabeled), there exists $y_0 \in \bar{B}_2$ such that

$$I_{v_0}(y_0,1) - I_{v_0}\big(y_0, 1/2\big) = 0 \quad \text{and} \quad y_0 \notin \mathcal{T}_\tau(V). \tag{5.15}$$

Proposition 2.7 implies that $I_{v_0}(y_0,r) = I_{v_0}(y_0,0^+)$ for every $r>0$ and by Lemma 5.2 we must have $y_0 \in S(v_0)$, thus contradicting $S(v_0) \subseteq V$ and $y_0 \notin \mathcal{T}_\tau(V)$. □

6. The measure of the free boundary

In this section we prove Theorem 1.1 that provides a local estimate of the Minkowski content, and thus of the Hausdorff measure, of the free boundary in the lower dimensional obstacle problem. Here we use a version of the inductive covering argument in [31, Section 8]. The key monotone quantity we consider is the maximal function of the frequency

$$\Theta_u(x,\rho) := \sup_{y\in\bar{B}_\rho(x)\cap\Gamma(u)} I_u(y,\rho). \tag{6.1}$$

Theorem 1.1 is a direct consequence of the following proposition.

Proposition 6.1. *For every $L>0$, there exists a constant $C_{6.1}(L)>0$ with this property: for any solution $u \not\equiv 0$ to the thin obstacle problem (1.1) in $B_{2\rho}(z) \subset \mathbb{R}^{n+1}$ with $z \in \mathbb{R}^n \times \{0\}$, we have*

$$\Theta_u(z,\rho) \leq L \quad \Longrightarrow \quad \mathcal{L}^{n+1}\Big(\mathcal{T}_r\big(\Gamma(u)\cap B_{\rho/2}(z)\big)\Big) \leq C_{6.1}(L)\, r^2\,\rho^{n-1} \quad \forall\, r\in(0,\rho). \tag{6.2}$$

Proof of Theorem 1.1. We are given u a solution to the lower dimensional obstacle problem in B_1 and $K \subset\subset B_1$. Set $\delta := 4^{-1}\mathrm{dist}(K,\partial B_1)$, let $\{B_\delta(x_i)\}_{i\in J}$, with J a (finite) maximal subset of points in $\Gamma(u)\cap K$, having mutual distance at least δ. Set $L := \max_{i\in J}\Theta_u(x_i,2\delta)$. Then, by applying Proposition 6.1 to every $B_{2\delta}(x_i)$, we have that

$$\begin{aligned}
\mathcal{L}^{n+1}\Big(\mathcal{T}_r\big(\Gamma(u)\cap K\big)\Big) &\leq \sum_{i\in J}\mathcal{L}^{n+1}\Big(\mathcal{T}_r\big(\Gamma(u)\cap B_\delta(x_i)\big)\Big)\\
&\leq \mathcal{H}^0(J)\, C_{6.1}(L)\,\delta^{n-1}\,r^2 =: C\,r^2 \qquad \forall\, r\in(0,2\delta).
\end{aligned}$$

We point out that the constant C depends only n, on $I_u(1)$ and on $\mathrm{dist}(K,\partial B_1)$. Indeed, L depends on $I_u(1)$ via Lemma 3.1; and since the balls $B_{\delta/2}(x_i)$ are disjoint, contained in B_1 and with centers in B_1', we can estimate $\mathcal{H}^0(J) \leq (2/\delta)^n$. □

The rest of the section is devoted to the proof of Proposition 6.1.

6.1. **Proof of Proposition 6.1.** By rescaling it is enough to consider the case $z = 0$ and $\rho = 1$. We start off with the case of minimal frequency $L = 1+s$: then, by Corollary 2.12 $\Theta_u(0,1) = 1+s$ and thus $u \in \mathcal{H}_{1+s}$ (cf. (5.6)). In turn, this implies that $\Gamma(u)$ is a $(n-1)$-dimensional hyperplane of $\{x_{n+1} = 0\}$ and (6.2) follows at once.

The proof is then completed by showing that $\sup \mathcal{S} = +\infty$ where

$$\mathcal{S} := \{L \in \mathbb{R} : \text{ Proposition 6.1 holds for } L\}.$$

The latter claim is in turn implied by the following fact: for every $L_0 > 1+s$ there exists a constant $\eta(L_0) > 0$ such that, if $L \in \mathcal{S}$ and $L < L_0$ then $L + \eta(L_0) \in \mathcal{S}$. In order to specify $\eta(L_0)$ we need to introduce several dimensional constants; to show the consistency of their choices, we declare them at the beginning (the readers can skip this list and refer to it each time the constants are introduced):

- $C_{6.2} := 10^{3(n-1)} C_{6.3}(n)$, where $C_{6.3}(n) > 1$ is the dimensional constant of Theorem 6.3;
- $\lambda = \min\{10^{-3}, 16^{2-n}\, C_{6.2}^{-1}\}$;
- $\tau = \min\big\{\lambda^2, 10^{-20n}\, C_{4.2}(2L_0, 45)^{-2}\, C_{6.2}^{-2}\, \lambda^{4n}\, \delta_{6.3}^2(\lambda)\big\}$, where $C_{4.2}(2L_0, 45)$ is the constant in Proposition 4.2 corresponding to $R = 45$ and $A = 2L_0$, and $\delta_{6.3}(\lambda)$ is the constant introduced in Theorem 6.3;
- $0 < \eta \leq \min\big\{\eta_{5.6}(\tau, 2L_0), \tau, L_0\big\}$, where $\eta_{5.6}(\tau, 2L_0)$ is the constant introduced in Proposition 5.6 with parameters, τ and $2L_0$.

Note that, for ever $L < L_0$ we have that $L + \eta \leq 2L_0$ with such a choice of η.

Then, the proof of Proposition 6.1 consists in showing that (6.2) holds for $L + \eta$, supposing that it has been verified for $L < L_0$. We proceed in several steps.

1. Let u be a solution in B_2 of the lower dimensional obstacle problem with $\Theta_u(0,1) \leq L + \eta$, and let $r \in (0,1)$ be the size of the tubular neighborhood in (6.2) (recall that $\rho = 1$ by scaling). For every $x \in \Gamma(u) \cap B_{1/2}$ we set $s_x := \max\big\{r_x, 2r\big\}$ with

$$r_x := \begin{cases} \inf\big\{t \in (0,1] \ : \ \Theta_u(x,t) > L\big\} & \text{if } I_u(x,1) > L, \\ 1 & \text{if } I_u(x,1) \leq L. \end{cases}$$

By definition, if $I_u(x,1) > L$, then

$$\exists\, y_x \in \bar{B}_{s_x}(x) \cap \Gamma(u) \quad \text{such that} \quad I_u(y_x, s_x) \geq L. \tag{6.3}$$

Let now $\{x_i\}_{i \in J} \subset \Gamma(u) \cap B_{1/2}$ be a finite collection of points such that the balls $B_{s_{x_i}/2}(x_i)$ constitute a Vitali covering of $\Gamma(u) \cap B_{1/2}$: *i.e.*

$$B_{s_{x_i}/10}(x_i) \cap B_{s_{x_j}/10}(x_j) = \emptyset \quad \forall\, i \neq j \in J \quad \text{and} \quad \Gamma(u) \cap B_{1/2} \subset \bigcup_{i \in J} B_{s_{x_i}/2}(x_i). \tag{6.4}$$

By construction, we have that

(i) $\mathcal{T}_r\big(\Gamma(u) \cap B_{s_{x_i}/2}(x_i)\big) \subseteq B_{s_{x_i}}(x_i)$, for all $i \in J$ because $2r \leq s_{x_i}$,
(ii) $\Theta_u(x_i, s_{x_i}) = L$ if $s_{x_i} > 2r$.

The key estimate is to show that there exists a dimensional constant $\bar{C} > 0$ such that

$$\sum_{i \in J} s_{x_i}^{n-1} \leq \bar{C}. \tag{6.5}$$

Indeed, assuming momentarily (6.5) we can prove (6.2) for $L+\eta$ as follows:

$$\begin{aligned}
\mathcal{L}^{n+1}\Big(\mathcal{T}_r\big(\Gamma(u)\cap B_{1/2}\big)\Big) &\overset{(6.4)}{\leq} \sum_{i\in J}\mathcal{L}^{n+1}\Big(\mathcal{T}_r(\Gamma(u)\cap B_{s_{x_i}/2}(x_i))\Big)\\
&\overset{\text{(i)}}{\leq} \sum_{i\in J\,:\,s_{x_i}>2r}\mathcal{L}^{n+1}\Big(\mathcal{T}_r(\Gamma(u)\cap B_{s_{x_i}/2}(x_i))\Big)+\sum_{i\in J\,:\,s_{x_i}=2r}\mathcal{L}^{n+1}\big(B_{s_{x_i}}(x_i)\big)\\
&\leq \sum_{i\in J\,:\,s_{x_i}>2r} C_{6.1}(L)\,r^2\,s_{x_i}^{n-1}+\sum_{i\in J\,:\,s_{x_i}=2r}\omega_{n+1}\,(2r)^{n+1}\\
&\leq \big(C_{6.1}(L)+2^{n+1}\omega_{n+1}\big)\;r^2\sum_{i\in J}s_{x_i}^{n-1}\overset{(6.5)}{\leq} C_{6.1}(L+\eta)\,r^2,
\end{aligned}$$

with $C_{6.1}(L+\eta) := \big(C_{6.1}(L)+2^{n+1}\omega_{n+1}\big)\,\bar{C}$. In the third inequality, we have used (6.2) itself with bound L on $\Theta_u(x_i,s_{x_i})$ in view of (ii).

2. Next we want to prove the claim (6.5). Let λ be the constant introduced at the beginning, we consider a suitable decomposition of the sets of centers $\{x_i\}_{i\in J}$:

$$\{x_i\}_{i\in J}=A^{(0)}\cup\bigcup_{l=1}^{N(\lambda)}A^{(l)},$$

with $A^{(0)} := \big\{x_i,\, i\in J\;:\;s_{x_i}\geq\lambda^2\big\}$, $N(\lambda)=\lfloor 10^n\lambda^{-3n}\rfloor+1$ and $A^{(l)}$ satisfying the following condition for $l>0$:

$$\forall\, x_i,x_j\in A^{(l)},\;\; x_i\in B_{s_{x_j}/\lambda}(x_j)\quad\Longrightarrow\quad s_{x_i}\leq\lambda^2\,s_{x_j}. \tag{6.6}$$

To see that such a decomposition exists, we follow [31, Lemma 5.4] and proceed inductively. We order the points in $A\setminus A^{(0)}$ according to a decreasing order of the corresponding radii: *i.e.*, $A\setminus A^{(0)}=\{p_i\}_{i\in J'}$ with $s_{p_{i+1}}\leq s_{p_i}$. Then, $p_1\in A^{(1)}$ and, if $p_1,\ldots,p_{i-1}$ have been sorted out, we assign p_i to some $A^{(l)}$ so that $A^{(l)}$ does not contain any point p_j with $j\leq i-1$, for which

$$p_i\in B_{s_{p_j}/\lambda}(p_j)\quad\text{and}\quad\lambda^2 s_{p_j}\leq s_{p_i}\leq s_{p_j}. \tag{6.7}$$

Note that for every j satisfying (6.7) we have $|p_i-p_j|\leq s_{p_j}/\lambda\leq s_{p_i}/\lambda^3$, thus $p_j\in B_{s_{p_i}/\lambda^3}(p_i)$; moreover, the balls $B_{s_{p_i}/10}(p_j)$ with j as in (6.7) are disjoint, as $s_{p_i}\leq s_{p_j}$ and $B_{s_{p_j}/10}(p_j)$ are disjoint by construction. Therefore, since $N(\lambda)$ is strictly bigger than the number of disjoint balls with radius $1/10$ in B_{1/λ^3} and center on B'_{1/λ^3}, one can surely find l so that $p_j\notin A^{(l)}$ for all j as in (6.7).

Let us check that (6.6) holds. Indeed, if $j<i$ (*i.e.* $s_{p_i}\leq s_{p_j}$) and $p_i\in B_{s_{p_j}/\lambda}(p_j)$, then the second condition in (6.7) must fail (being $p_j\in A^{(l)}$), *i.e.* $s_{p_i}<\lambda^2\,s_{p_j}$. On the other hand, if $i<j$ (*i.e.* $s_{p_j}\leq s_{p_i}$), from $p_i\in B_{s_{p_j}/\lambda}(p_j)$, we deduce $p_j\in B_{s_{p_j}/\lambda}(p_i)\subset B_{s_{p_i}/\lambda}(p_i)$ and, as $p_j\in A^{(l)}$, the second condition in (6.7) must fail, *i.e.* $s_{p_j}<\lambda^2\,s_{p_i}$. But this is a contradiction because $\lambda<1/10$ and

$$\frac{s_{p_i}}{10}\leq|p_i-p_j|\leq\frac{s_{p_j}}{\lambda}<\lambda\,s_{p_i}.$$

3. Next, for $l\in\{0,\ldots,N(\lambda)\}$ we introduce the measures:

$$\mu^l := \sum_{x\in A^{(l)}}s_x^{n-1}\delta_x. \tag{6.8}$$

To conclude (6.5), we show that there exists a dimensional constant $C_0>0$ such that

$$\mu^l\big(B_{1/2}\big)\leq C_0\quad\forall\, l\in\{0,\ldots,N(\lambda)\}. \tag{6.9}$$

Indeed, from (6.9) we infer (6.5) with the constant $\bar{C} := \big(N(\lambda)+1\big)\,C_0$:

$$\sum_{i\in J}s_{x_i}^{n-1}\leq\sum_{l=0}^{N(\lambda)}\mu^l(B_{1/2})\leq\big(N(\lambda)+1\big)\,C_0=\bar{C}.$$

The case $l = 0$ is straightforward: since the balls $B_{\lambda^2/10}(x)$ with $x \in A^{(0)}$ are pairwise disjoint, contained in B_1 and with center $x \in B'_{1/2}$, then $\mathcal{H}^0\big(A^{(0)}\big) \leq \frac{10^n}{\lambda^{2n}}$. Being $s_x \leq 2$ we deduce that

$$\mu^0(B_{1/2}) = \sum_{x \in A^{(0)}} s_x^{n-1} \leq \mathcal{H}^0\big(A^{(0)}\big) 2^{n-1} \leq \frac{20^n}{\lambda^{2n}},$$

and estimate (6.9) for $l = 0$ follows as soon as $C_0 \geq 20^n/\lambda^{2n}$.

For the remaining cases, we are going to show the following lemma.

Lemma 6.2. *Let μ^l be the measures in (6.8) with $l \geq 1$. Then,*

$$\mu^l\big(B_\rho(x)\big) \leq C_{6.2}\, \rho^{n-1}, \tag{6.10}$$

for every $x \in \operatorname{spt}(\mu^l)$ and for every $\rho \in (s_x, \lambda^2]$, where $C_{6.2} > 0$ is the dimensional constant introduced at the beginning.

Lemma 6.2 implies (6.9). Indeed, let us consider a maximal subset of points $\{x_i\}_{i \in J^{(l)}} \subseteq A^{(l)}$ with $|x_i - x_j| \geq \lambda^2$ for all $i \neq j \in J^{(l)}$. Then, the balls $\{B_{\lambda^2/2}(x_i)\}_{i \in J^{(l)}}$ are disjoint, contained in B_1 (as $\lambda < 1/2$), and with centers $x_i \in B'_{1/2}$. Thus, $\mathcal{H}^0(J^{(l)}) \leq 2^n/\lambda^{2n}$ and by maximality of the number of points in $\{x_i\}_{i \in J^{(l)}}$ we have also $\operatorname{spt}(\mu^l) \subset \cup_{i \in J^{(l)}} B_{\lambda^2}(x_i)$. Then

$$\mu^l(B_{1/2}) \leq \sum_{i \in J^{(l)}} \mu^l\big(B_{\lambda^2}(x_i)\big) \overset{(6.10)}{\leq} \mathcal{H}^0(J^{(l)})\, C_{6.2} \lambda^{2(n-1)} \leq \frac{2^n}{\lambda^2} C_{6.2},$$

and (6.9) follows with $C_0 := \max\big\{2^n\, C_{6.2}/\lambda^2, 20^n/\lambda^{2n}\big\}$.

Proposition 6.1, and hence Theorem 1.1, are now established once we show Lemma 6.2.

6.2. **Proof of Lemma 6.2.** We fix $l \in \{1, \ldots, N(\lambda)\}$, and set $s_{\min} := \min\big\{s_w \;:\; w \in A^{(l)}\big\}$, $k_{\max} := \lfloor \log_\lambda(s_{\min}) \rfloor$. Note that $k_{\max} \geq 2$ and $\lambda^{k_{\max}+1} < s_{\min} \leq \lambda^{k_{\max}}$. We prove (6.10) for all $\rho \in (\max\{\lambda^{k+1}, s_{\min}\}, \lambda^k]$ by induction on $k \in \{2, \ldots, k_{\max}\}$ in decreasing order. More precisely, the base induction step is for $k = k_{\max}$. In this case, for every point $w_0 \in \operatorname{spt}(\mu^l) = A^{(l)}$ with $s_{w_0} \leq \lambda^{k_{\max}}$ we have that $\operatorname{spt}(\mu^l) \cap B_{\lambda^{k_{\max}}}(w_0) = \{w_0\}$, from which (6.10) readily follows. Indeed, if $w \in B_{\lambda^{k_{\max}}}(w_0) \cap A^{(l)}$ is different from w_0, then $w \in B_{s_{w_0}/\lambda}(w_0) \cap A^{(l)}$ as $s_{w_0} \in (\lambda^{k_{\max}+1}, \lambda^{k_{\max}}]$ and by (6.6) we reach a contradiction

$$\lambda^{k_{\max}+1} < s_{\min} \leq s_w \overset{(6.6)}{\leq} \lambda^2 s_{w_0} \leq \lambda^{k_{\max}+2}.$$

We can then proceed inductively: we assume that we have shown (6.10) for every $\rho \in (s_{\min}, \lambda^{k+1}]$ for some $k \in \{2, \ldots, k_{\max} - 1\}$ and for all $w \in \operatorname{spt}(\mu^l)$ with $s_w \leq \lambda^{k+1}$, and then we prove that

$$\mu^l\big(B_t(w_0)\big) \leq C_{6.2}\, t^{n-1} \quad \forall\, t \in (\lambda^{k+1}, \lambda^k], \quad \forall\, w_0 \in \operatorname{spt}(\mu^l) \text{ with } s_{w_0} < t. \tag{6.11}$$

1. Let $t \in (\lambda^{k+1}, \lambda^k]$ with $k \geq 2$ and $w_0 \in \operatorname{spt}(\mu^l)$ be such that $s_{w_0} < t$. We set $W := \operatorname{spt}(\mu^l) \cap B_t(w_0)$ and

$$W^{(1)} := \big\{w \in W \;:\; I_u(w, s_w) < L - \tau\big\} \quad \text{and} \quad W^{(2)} := W \setminus W^{(1)},$$

where τ is the constant introduced at the beginning. Next we order the points in $W^{(1)}$ in such a way that $W^{(1)} = \{p_h\}_h$ with $s_{p_h} \geq s_{p_{h+1}}$; and we define inductively $z_1 := y_{p_1}$ and for $\alpha \geq 2$

$$z_\alpha := y_{p_{m_\alpha}} \text{ with } m_\alpha = \min\Big\{h \;:\; y_{p_h} \notin \cup_{j=1}^{\alpha-1} B_{s_{z_j}}(z_j)\Big\}, \tag{6.12}$$

where the y_p's are the points defined in (6.3) (which exist because $p_h \in A^{(l)}$ with $l \geq 1$ implies $s_{p_h} < \lambda^2 < 1$, *i.e.* $I_u(p_h, 1) > L$). Let Z be the set of the selected points z_α's and set $s_{z_\alpha} := s_{p_{m_\alpha}}$ (with a slight abuse of notation), $E := \cup_{z_\alpha \in Z} B_{s_{z_\alpha}}(z_\alpha)$ and

$$\mu_1^l := \sum_{z_\alpha \in Z} s_{z_\alpha}^{n-1}\, \delta_{z_\alpha} + \sum_{w \in W^{(2)} \setminus E} s_w^{n-1} \delta_w.$$

The measure μ_1^l satisfies the following five properties:

$$\forall\, p \in \mathrm{spt}\,(\mu_1^l) \quad \Delta^1_{s_p}(p) = I_u(p,1) - I_u(p,s_p) \le \eta + \tau \le 2\,\tau, \tag{6.13}$$

$$\forall\, p \neq p' \in \mathrm{spt}\,(\mu_1^l) \quad \max\{s_p, s_{p'}\} \le 20\,|p-p'|, \tag{6.14}$$

$$\mathrm{spt}\,(\mu_1^l) \subset B_{11t}(w_0), \tag{6.15}$$

$$\mu^l\big(B_t(w_0)\big) \le 2\,\mu_1^l(B_{11t}(w_0)), \tag{6.16}$$

$$\mu_1^l\big(B_\rho(p)\big) \le 10^{12n}\,\lambda^{-2n}\,C_{6.2}\,\rho^{n-1} \quad \forall\, p \in \mathrm{spt}\,(\mu_1^l),\ \forall\, \rho \in [{}^{s_p}\!/20, 10^2\lambda^k]. \tag{6.17}$$

The properties (6.13) and (6.14) follows directly from the definition of μ_1^l. More precisely, for (6.13) recall the choice $\eta \le \tau$ and that by assumption $\Theta_u(0,1) \le L+\eta$. Therefore, the conclusion follows either by (6.3) if $p \in Z$ or otherwise by the very definition of $W^{(2)}$.

For (6.14) we distinguish three cases:

(i) $p,\, p' \in Z$. Assume without loss of generality that $s_p \le s_{p'}$, then by the selection procedure defining Z itself $p \notin B_{s_{p'}}(p')$, and thus $s_{p'} < |p-p'|$;

(ii) $p \in Z$, $p' \in W^{(2)} \setminus E$. Then $p' \notin B_{s_p}(p)$ by definition of E, so that $s_p < |p-p'|$. Moreover, if $p = y_w$, with $w \in W^{(1)}$, we use (6.4) to infer

$$s_{p'} < 10|w-p'| \le 10(|w-y_w| + |y_w - p'|) \le 10(s_p + |p-p'|) \le 20|p-p'|.$$

(iii) $p,\, p' \in W^{(2)} \setminus E$. Since $B_{\frac{s_p}{10}}(p) \cap B_{\frac{s_{p'}}{10}}(p') = \emptyset$ by (6.4), $\max\{s_p, s_{p'}\} < 10|p-p'|$.

For what concerns (6.15), we notice that for all $w \in W$ by (6.4) we have that $s_w < 10|w-w_0| \le 10t$, and therefore

$$|y_w - w_0| \le |y_w - w| + |w-w_0| \le s_w + |w-w_0| \le 11|w-w_0| \le 11t.$$

Eq. (6.16) and (6.17) are proven in the next two steps. The proof of (6.11) will then be a consequence of (6.13) – (6.17) only and it will be detailed in step 4.

2. For what concerns (6.16), for every $z_\alpha \in Z$ we introduce the sets

$$W^{z_\alpha} := \Big(W^{(2)} \cap \big(B_{s_{z_\alpha}}(z_\alpha) \setminus \cup_{j=1}^{\alpha-1} B_{s_{z_j}}(z_j)\big)\Big) \cup \Big\{w \in W^{(1)} \,:\, y_w \in B_{s_{z_\alpha}}(z_\alpha) \setminus \cup_{j=1}^{\alpha-1} B_{s_{z_j}}(z_j)\Big\}.$$

Hence, as by the very definition of E

$$W^{(2)} \cap E = \cup_\alpha \Big(W^{(2)} \cap \big(B_{s_{z_\alpha}}(z_\alpha) \setminus \cup_{j=1}^{\alpha-1} B_{s_{z_j}}(z_j)\big)\Big),$$

and by that of Z

$$W^{(1)} = \cup_\alpha \Big\{w \in W^{(1)} \,:\, y_w \in B_{s_{z_\alpha}}(z_\alpha) \setminus \cup_{j=1}^{\alpha-1} B_{s_{z_j}}(z_j)\Big\},$$

then $W = \cup_{z\in Z} W^z \cup (W^{(2)} \setminus E)$ and

$$\mu^l(B_t(w_0)) = \sum_{z\in Z} \mu^l(W^z) + \mu^l\big(W^{(2)} \setminus E\big) = \sum_{z\in Z} \mu^l(W^z) + \mu_1^l\big(W^{(2)} \setminus E\big). \tag{6.18}$$

We will prove (6.16) by showing that, for every $z_\alpha \in Z$, we have

$$\mu^l(W^{z_\alpha}) \le 2 s_{z_\alpha}^{n-1}. \tag{6.19}$$

Indeed, from (6.19) we immediately deduce that

$$\begin{aligned}\mu^l(B_t(w_0)) &\overset{(6.18)}{=} \sum_{z_\alpha\in Z} \mu^l(W^{z_\alpha}) + \mu_1^l\big(W^{(2)} \setminus E\big) \overset{(6.19)}{\le} 2\sum_{z_\alpha \in Z} s_{z_\alpha}^{n-1} + \mu_1^l\big(W^{(2)} \setminus E\big) \\ &\le 2\,\mu_1^l(B_{11t}(w_0)).\end{aligned}$$

The key observation to establish (6.19) is the following: let $\bar w \in W^{(1)}$ be such that $z_\alpha = y_{\bar w}$. Then, by definition

$$I(z_\alpha, s_{\bar w}) - I(\bar w, s_{\bar w}) > L - L + \tau = \tau.$$

We can then apply Proposition 5.6 in $B_{8s_{\bar w}}(z_\alpha)$ with parameters τ, $2L_0$. Indeed, $I_u(z_\alpha, 8s_{\bar w}) \le \Theta_u(0,1) \le L + \eta \le 2L_0$ (recall that $L < L_0$ and we have chosen $\eta \le L_0$). Moreover, as we have imposed $\eta \leq \eta_{5.6}(\tau, 2L_0)$, we deduce that the first case of the dichotomy of Proposition 5.6 does not occur: *i.e.* there exists a $(n-2)$-dimensional affine subspace passing through z_α such that

$$\forall q \in \Gamma(u) \cap B_{4s_{\bar w}}(z_\alpha) \quad \text{with} \quad \Delta^{2s_{\bar w}}_{s_{\bar w}}(q) \leq \eta \quad \Longrightarrow \quad q \in \mathcal{T}_{2\tau s_{\bar w}}(V). \tag{6.20}$$

Eq. (6.20) is the main ingredient of the proof, because it implies that all the points in W^{z_α} different from $\bar w$ have clustered around a lower dimensional space V, namely

$$W^{z_\alpha} \setminus \{\bar w\} \subseteq \mathcal{T}_{4\lambda^2 s_{\bar w}}(V). \tag{6.21}$$

Indeed, consider a generic point $w \in W^{z_\alpha} \setminus \{\bar w\}$. If $w \in W^{(2)} \cap B_{s_{z_\alpha}}(z_\alpha)$, then $w \in B_{2s_{\bar w}}(\bar w)$ and by (6.6) we have $s_w \leq \lambda^2 s_{\bar w}$. In turn this implies that $y_w \in B_{s_w}(w) \subset B_{2s_{z_\alpha}}(z_\alpha)$ and

$$\Delta^{2s_{\bar w}}_{s_{\bar w}}(y_w) \leq I(y_w, 1) - I(y_w, s_w) \leq L + \eta - L = \eta. \tag{6.22}$$

Therefore, by (6.20) we infer that $y_w \in \mathcal{T}_{2\tau s_{\bar w}}(V)$ and, since $\tau \leq \lambda^2$, also $w \in \mathcal{T}_{2\tau s_{\bar w}+s_w}(V) \subset \mathcal{T}_{4\lambda^2 s_{\bar w}}(V)$. On the other hand, if $w \in W^{(1)}$, then by the selection procedure (recall the decreasing order of the radii s_{z_j}), we have that $s_w \leq s_{\bar w}$: in particular $w \in B_{s_{\bar w}}(y_w) \subset B_{3s_{\bar w}}(\bar w)$. Therefore, thanks to (6.6) we have also $s_w \leq \lambda^2 s_{\bar w}$ and (6.22) holds. By (6.20) $y_w \in \mathcal{T}_{2\tau s_{\bar w}}(V)$ and hence $w \in \mathcal{T}_{4\lambda^2 s_{\bar w}}(V)$, thus showing (6.21).

Then the proof of (6.19) follows from an elementary covering argument. Let $Q' \subset W^{z_\alpha} \setminus \{\bar w\}$ be a maximal collection of points such that the balls $\{B_{\lambda s_{\bar w}/20}(p)\}_{p\in Q'}$ are pairwise disjoint: in particular $W^{z_\alpha} \setminus \{\bar w\} \subset \cup_{p\in Q'} B_{\lambda s_{\bar w}/10}(p)$. Let $\pi_V : \mathbb{R}^n \to V$ be the nearest point projection on V and note that, since $\lambda \leq 1/160$, we have

$$B_{\lambda s_{\bar w}/40}(\pi_V(p)) \subset B_{\lambda s_{\bar w}/20}(p) \subset B_{4s_{\bar w}}(z_\alpha), \quad \forall\, p \in Q',$$

where we used that every $p \in W^{z_\alpha}$ is contained in $B_{3s_{\bar w}}(z_\alpha)$, and thus $p \in B_{4s_{\bar w}}(\bar w)$. Therefore, $B_{\lambda s_{\bar w}/40}(\pi_V(p)) \cap V$ are pairwise disjoint for $p \in Q'$ and contained in $B_{4s_{\bar w}}(\pi_V(z_\alpha)) \cap V$. This allows us to give an estimate on the cardinality of Q', namely $\mathcal{H}^0(Q') \leq 160^{n-2}/\lambda^{n-2}$. In proving the latter estimate we have crucially used that V has dimension $n-2$. Now by the inductive hypothesis (6.11) we get (6.19):

$$\begin{aligned}\mu^l(W^{z_\alpha}) &\leq \mu^l(\{\bar w\}) + \sum_{p\in Q'} \mu^l\big(B_{\lambda s_{\bar w}/10}(p)\big) \leq s_{\bar w}^{n-1} + \mathcal{H}^0(Q')\, C_{6.2}\, (\lambda/10)^{n-1}\, s_{\bar w}^{n-1}\\ &\leq s_{\bar w}^{n-1} + 16^{n-2}\lambda\, C_{6.2}\, s_{\bar w}^{n-1} \leq 2\, s_{\bar w}^{n-1} = 2\, s_{z_\alpha}^{n-1},\end{aligned}$$

thanks to the choice $\lambda < 16^{2-n} C_{6.2}^{-1}$. We can apply the inductive hypothesis to $B_{\lambda s_{\bar w}/10}(p)$ since $s_p \leq \lambda^2 s_{\bar w} < \lambda s_{\bar w}/10 \leq \lambda^{k+1}$ (the first inequality holds thanks to (6.6) because for every $p \in Q'$ we have that $p \in B_{4s_{\bar w}}(\bar w)$, and the last one in view of $s_w \leq 10\, t \leq 10\, \lambda^k$ for every $w \in W$).

3. We show next (6.17). Let p, ρ be as in the statement. For every $q \in \operatorname{spt}(\mu^l_1) \cap B_\rho(p)$ let $x_q \in \operatorname{spt}(\mu^l)$ be a point such that $y_{x_q} = q$ if $q \notin \operatorname{spt}(\mu^l)$ and coinciding with q itself otherwise. Then,

$$|x_p - x_q| \leq |x_p - p| + |p - q| + |q - x_q| \leq s_p + \rho + s_q \overset{(6.14)}{\leq} \rho + 40|q-p| < 41\rho.$$

Therefore, for every point $q \in \operatorname{spt}(\mu^l_1) \cap B_\rho(p)$ we have that the corresponding point x_q belongs to $\operatorname{spt}(\mu^l) \cap B_{41\rho}(x_p)$, so that

$$\mu^l_1\big(B_\rho(p)\big) \leq \mu^l\big(B_{41\rho}(x_p)\big).$$

The proof of (6.17) is now a consequence of the inductive hypothesis (6.11) and a covering argument. Indeed,

(i) if $41\rho \leq \lambda^{k+1}$: we can apply (6.10) directly (since $s_p \leq 20\rho$ by assumption), and infer that $\mu^l\big(B_\rho(p)\big) \leq C_{6.2} 41^{n-1}\rho^{n-1}$;

(ii) if $41\rho > \lambda^{k+1}$: we cover $\operatorname{spt}(\mu^l) \cap B_{41\rho}(x_p)$ with balls $B_{\lambda^{k+1}/10}(w)$ having centers $w \in \operatorname{spt}(\mu^l)$ such that half the balls are disjoint. Since $\rho \leq 10^2\lambda^k$ by assumption (cf. (6.17)) and the centers are in B'_1, the cardinality of the cover can be estimated by $(10^5/\lambda)^n$. Moreover, $s_w \leq 20 \cdot 41\rho \leq 10^5\, \lambda^k$ in view of $w \in B_{41\rho}(x_p)$ (cf. (6.4)), and $\rho \leq 10^2\lambda^k$ by assumption (cf. (6.17)) . Hence, in case $s_w \leq \lambda^{k+1}$ we can use the inductive hypothesis (6.11) to infer that $\mu^l(B_{\lambda^{k+1}/10}(w)) \leq C_{6.2}\lambda^{(k+1)(n-1)}$. Otherwise, if $s_w > \lambda^{k+1}$, $\mathrm{spt}\,(\mu^l) \cap B_{\lambda^{k+1}/10}(w) = \{w\}$ by (6.4), and thus $\mu^l(B_{\lambda^{k+1}/10}(w)) = s_w^{n-1} \leq 10^{5(n-1)}\,\lambda^{k(n-1)}$. In conclusion, recalling that $\lambda^{k+1} \leq 41\rho$, we infer that

$$\begin{aligned}\mu_1^l\big(B_\rho(p)\big) &\leq \mu^l\big(B_{41\rho}(x_p)\big) \leq (10^5/\lambda)^n 10^{5(n-1)}\; C_{6.2}\lambda^{k(n-1)}\\ &\leq 10^{12n}\,\lambda^{-2n}\,C_{6.2}\,\rho^{n-1}.\end{aligned}$$

4. We are now in the position to infer (6.11) from (6.13)–(6.17), thus concluding the proof of Lemma 6.2. We start off estimating the generalized Jones' number for μ_1^l (for simplicity we omit the subscripts in their notation): for every $\rho \in (0, 44t]$, with $t \in (\lambda^{k+1}, \lambda^k]$ by (6.11), and $w \in \mathrm{spt}\,(\mu_1^l)$, using Proposition 4.2 with parameters $A = 2L_0$ and $R = 45$ (recall that $L_0+\eta \leq 2L_0$ and do not confuse the radius R there with the one in this proof) we infer

$$\beta^2(w,\rho) \leq \frac{C_{4.2}}{\rho^{n-1}} \int_{B_\rho(w)} \Delta_{20\rho}^{94\rho}(z)\,\chi_{[0,20\rho]}(s_z)\,\mathrm{d}\mu_1^l(z), \tag{6.23}$$

having used that $s_z \leq 20\rho$ if $z \in \mathrm{spt}\,(\mu_1^l) \cap B_\rho(w)$ by (6.14). Integrating (6.23) over $B_R(\bar w)$ for $\bar w \in \mathrm{spt}\,(\mu_1^l)$, with $R \in (0, 44\lambda^k]$ and $\rho \in (0, R]$, we get

$$\begin{aligned}\int_{B_R(\bar w)} \beta^2\big(w,\rho\big)\,\mathrm{d}\mu_1^l(w) &\overset{(6.23)}{\leq} \frac{C_{4.2}}{\rho^{n-1}} \int_{B_R(\bar w)}\int_{B_\rho(w)} \Delta_{20\rho}^{94\rho}(z)\,\chi_{[0,20\rho]}(s_z)\,\mathrm{d}\mu_1^l(z)\,\mathrm{d}\mu_1^l(w)\\ &\leq \frac{C_{4.2}}{\rho^{n-1}} \int_{B_{R+\rho}(\bar w)} \mu_1^l\big(B_\rho(z)\big)\Delta_{20\rho}^{94\rho}(z)\,\chi_{[0,20\rho]}(s_z)\,\mathrm{d}\mu_1^l(z)\\ &\overset{(6.17)}{\leq} \bar C(\lambda)\int_{B_{2R}(\bar w)} \Delta_{20\rho}^{94\rho}(z)\,\chi_{[0,20\rho]}(s_z)\,\mathrm{d}\mu_1^l(z).\end{aligned} \tag{6.24}$$

In the second inequality we have used Fubini's theorem, and we have set for simplicity $\bar C(\lambda) := C_{4.2}\,C_{6.2}\,{}^{10^{12n}}/_{\lambda^{2n}}$. Let us now introduce the following notation for the average oscillation of a measure μ at scale λ on the ball $B_\varrho(\bar w)$:

$$\mathrm{Osc}_\mu^\lambda(\bar w, \varrho) := \int_{B_\varrho(\bar w)} \sum_{j=0}^{+\infty} \beta^2(y, \lambda^j\,\varrho)\,\mathrm{d}\mu(y).$$

Then, summing (6.24) for $\rho = \lambda^j R$ with $j \in \mathbb{N}$ and using $\lambda \leq 10^{-3}$, we get

$$\begin{aligned}\mathrm{Osc}_{\mu_1^l}^\lambda(\bar w, R) &\leq \bar C(\lambda)\int_{B_{2R}(\bar w)} \sum_{j=0}^{\lfloor \log_\lambda(s_z/20R)\rfloor} \Delta_{20\lambda^j R}^{20\lambda^{j-1}R}(z)\,\mathrm{d}\mu_1^l(z)\\ &\leq \bar C(\lambda)\int_{B_{2R}(\bar w)} \Delta_{s_z}^1(z)\,\mathrm{d}\mu_1^l(z) \overset{(6.13)}{\leq} 2\,\tau\,\bar C(\lambda)\,\mu_1^l(B_{2R}(\bar w))\\ &\overset{(6.17)}{\leq} 2^n\,\bar C(\lambda)^2\,\tau\,R^{n-1},\end{aligned} \tag{6.25}$$

by taking into account that $s_z/20 \leq \rho < R$ and $20\lambda^{-1}R < 1$ (being $R \leq 44\lambda^k$ and $k \geq 2$). In addition, we notice that in case $R < s_{\bar w}/20$, estimate (6.25) still holds true. Indeed, in such a case $B_R(\bar w) \cap \mathrm{spt}\,(\mu_1^l) = \{\bar w\}$ and by definition $\beta(\bar w, \lambda^j R) = 0$ for every $j \in \mathbb{N}$, so that $\mathrm{Osc}_{\mu_1^l}^\lambda(\bar w, R) = 0$

Note moreover that (6.25) can be extended to every ball $B_R(p)$ with $p \in B_{22t}(w_0)$ and $R \in (0, 22t]$: indeed, if $B_R(p) \cap \mathrm{spt}\,(\mu_1^l) = \emptyset$, then $\mathrm{Osc}_{\mu_1^l}^\lambda(p, R) = 0$; otherwise, if $w \in B_R(p) \cap \sup(\mu_1^l)$, then $B_R(p) \subset B_{2R}(w)$ and

$$\mathrm{Osc}_{\mu_1^l}^\lambda(p, R) \leq 2^{n+1}\mathrm{Osc}_{\mu_1^l}^\lambda(w, 2R) \overset{(6.25)}{\leq} 2^{2n+1}\,\bar C(\lambda)^2\,\tau\,R^{n-1}, \tag{6.26}$$

being $\beta(y,\rho) \leq 2^{n+1}\beta(y, 2\rho)$ for every y and every $\rho > 0$.

The conclusion of the proof is now an application of the following result by Naber and Valtorta [31, Theorem 3.4 & Remark 3.9].

Theorem 6.3 (Naber–Valtorta [31])**.** *There is a dimensional constant $C_{6.3}(n) > 0$ such that the following holds. For every $\lambda > 0$, there exists $\delta_{6.3}(\lambda) > 0$ with this property: for every $\{B_{r_i}(x_i)\}_{i\in I}$ finite collection of pairwise disjoint balls in $B_2 \subset \mathbb{R}^n$ and $\mu := \sum_{i\in I} r_i^{n-1}\delta_{x_i}$,*

$$\mathrm{Osc}^{\lambda}_{\mu}(x,\varrho) \le \delta^2_{6.3}(\lambda)\,\varrho^{n-1} \quad \forall\, B_\varrho(x) \subset B_2 \quad \Longrightarrow \quad \mu(B_1) \le C_{6.3}.$$

Renaming for simplicity the points in the support of μ^l_1 as $\mu^l_1 = \sum_i s_{p_i}^{n-1}\,\delta_{p_i}$, we can apply Theorem 6.3 with $x_i := (p_i - w_0)/(11t)$, $r_i := s_{p_i}/(440t)$, and $\mu := \sum_i r_i^{n-1}\delta_{x_i}$. Indeed, from (6.15) we have that $\mathrm{spt}\,(\mu) \subset B_1$ and from (6.14) we have that $B_{r_i}(x_i)$ are pairwise disjoint. Moreover, from (6.26) and the choice of τ it follows that, for every $B_r(x) \subset B_2$ we have

$$\begin{aligned}\mathrm{Osc}^{\lambda}_{\mu}(x,r) &= \frac{1}{(40)^{2(n-1)}(11t)^{n-1}}\,\mathrm{Osc}^{\lambda}_{\mu^l_1}(w_0 + 11tx, 11tr)\\ &\le \frac{2^{2n+1}}{(40)^{2(n-1)}(11t)^{n-1}}\,\bar{C}(\lambda)^2\,\tau\,(11tr)^{n-1} \le \delta^2_{6.3}(\lambda)\,r^{n-1}.\end{aligned}$$

We then conclude that $\mu^l_1(B_{11t}(w_0)) = (440t)^{n-1}\mu(B_1) \le C_{6.3}\,(440t)^{n-1}$ and, by the choice of the constant $C_{6.2}$, we conclude (6.10):

$$\mu^l(B_t(w_0)) \overset{(6.16)}{\le} 2\mu^l_1(B_{11t}(w_0)) \le 2\,C_{6.3}\,(440t)^{n-1} = C_{6.2}\,t^{n-1}.$$

7. Structure of the free boundary $\mathcal{H}^{n-1}$-a.e.

In this section we give the proof of Theorem 1.2. It is a consequence of Theorem 1.1 and of the following rectifiability criterion recently established by Azzam & Tolsa [6, Theorem 1.1]. A similar criterion has also been established independently by Naber & Valtorta in [31, Theorem 3.3].

7.1. **Azzam–Tolsa rectifiability criterion.** We recall the following definition: a Radon measure μ in $\mathbb{R}^n$ is called k-rectifiable if

(i) μ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^k$, *i.e.* for every $E \subset \mathbb{R}^n$

$$\mathcal{H}^k(E) = 0 \quad \Longrightarrow \quad \mu(E) = 0,$$

(ii) there exist at most countable many C^1 functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$, for $i \in \mathbb{N}$, such that

$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)\Big) = 0.$$

A set $E \subset \mathbb{R}^n$ is said $\mathcal{H}^k$-rectifiable if the associated measure $\mathcal{H}^k \llcorner E$ is k-rectifiable.

The following is the rectifiability criterion we are going to exploit: in order to state it, we need to recall the notion of upper-density of a measure

$$\vartheta^{k,\star}(x,\mu) := \limsup_{r\to 0^+} \frac{\mu\big(B_r(x)\big)}{\omega_k\, r^k}.$$

Theorem 7.1 (Azzam–Tolsa [6])**.** *Let μ be a finite Borel measure in $\mathbb{R}^n$ with $\vartheta^{k,\star}(x,\mu) < +\infty$ for μ-a.e. $x \in \mathbb{R}^n$. Then, μ is k-rectifiable if*

$$\int_0^1 \frac{\big(\beta^{(k)}_{\mu,2}(x,r)\big)^2}{r}\,\mathrm{d}r < +\infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n, \tag{7.1}$$

The following two remarks are in order.

Remark 7.2. In the case $E \subset \mathbb{R}^n$ is a Borel set with $\mathcal{H}^k(E) < +\infty$, then $\mu := \mathcal{H}^k \llcorner E$ has upper-density finite μ-almost everywhere. More precisely, $\vartheta^{k,\star}(x,\mu) \le 1$ for μ-a.e. $x \in E$ (see for instance [2, (2.43)]).

Remark 7.3. Let $\lambda \in (0,1)$ be any number. For every $\lambda^{q+1} \le r < \lambda^q$ (with $q \in \mathbb{N}$) we have that $\beta^{(k)}_{\mu,2}(x,r) \le C\,\beta^{(k)}_{\mu,2}(x,\lambda^q)$ for some constant $C = C(\lambda,k)$, and hence

$$\int_0^1 \frac{\big(\beta^{(k)}_{\mu,2}(x,r)\big)^2}{r}\,\mathrm{d}r = \sum_{q=0}^{\infty}\int_{\lambda^{q+1}}^{\lambda^q} \frac{\big(\beta^{(k)}_{\mu,2}(x,r)\big)^2}{r}\,\mathrm{d}r \le C\sum_{q=0}^{\infty}\big(\beta^{(k)}_{\mu,2}(x,\lambda^q)\big)^2. \tag{7.2}$$

We can now prove that $\Gamma(u)$ is $\mathcal{H}^{n-1}$-rectifiable.

7.2. **Proof of Theorem 1.2.** We are given a solution u to the lower dimensional obstacle problem in B_1 and we want to show that $\Gamma(u) \cap B_R$ is rectifiable for every $R < 1$. Set $\delta := {}^{(1-R)}/_2$ and let $\{B_\delta(x_i)\}_{i\in J}$ be a finite covering of $\Gamma(u) \cap B_R$, with $x_i \in \Gamma(u)$, and set $L := \max_{i\in J} \Theta_u(x_i, 2\delta)$. Then, it suffices to show that $\Gamma(u) \cap B_\delta(x_i)$ is rectifiable.

After a suitable change of variable ($v(x) := u(x_i + \delta\, x)$ – cf. Remark 2.6), we are left to verify the following statement: let v be a solution to the lower dimensional obstacle problem in B_2 with $0 \in \Gamma(v)$, then $\Gamma(v) \cap B_1$ is rectifiable. To this aim, for every $l \in \mathbb{N} \setminus \{0\}$ we consider the following sets:

$$E_l := \Big\{ x \in \Gamma(v) \cap B_1 \ : \ \mathcal{H}^{n-1}\big(\Gamma(v) \cap B_r(x)\big) \leq 2\,\omega_{n-1}\, r^{n-1} \quad \forall\, r \in \Big(0, \frac{1}{l}\Big) \Big\}. \tag{7.3}$$

Note that $E_l \subseteq E_{l+1}$; and that Theorem 1.1 and Remark 7.2 imply

$$\mathcal{H}^{n-1}\Big(\Gamma(v) \cap B_1 \setminus \cup_{l=1}^{\infty} E_l\Big) = 0. \tag{7.4}$$

Therefore, it is now enough to show that E_l is $\mathcal{H}^{n-1}$-rectifiable for any fixed integer $l \in \mathbb{N}$; in this respect we set $\mu_l := \mathcal{H}^{n-1} \llcorner E_l$. We fix $\lambda \in (0, {}^1/_{18})$ and an integer q_0 such that $\lambda^{q_0 - 1} \leq 1$. By applying Proposition 4.2 (with parameter $R = 7$) we have that

$$\begin{aligned}
\sum_{q=q_0}^{+\infty} \int_{B_1} \beta_{\mu_l}^2(y, \lambda^q)\, \mathrm{d}\mu_l(y) &\leq \sum_{q=q_0}^{+\infty} \frac{C_{4.2}}{\lambda^{q(n-1)}} \int_{B_1} \int_{B_{\lambda^q}(y)} \Delta_{\lambda^q}^{18\lambda^q}(x)\, \mathrm{d}\mu_l(x)\, \mathrm{d}\mu_l(y) \\
&\leq \sum_{q=q_0}^{+\infty} \frac{C_{4.2}}{\lambda^{q(n-1)}} \int_{B_{3/2}} \mu_l(B_{\lambda^q}(x))\, \Delta_{\lambda^q}^{18\lambda^q}(x)\, \mathrm{d}\mu_l(x) \\
&\overset{(7.3)}{\leq} 2\,\omega_{n-1}\, C_{4.2} \int_{B_{3/2}} \sum_{q=q_0}^{+\infty} \Delta_{\lambda^q}^{\lambda^{q-1}}(x)\, \mathrm{d}\mu_l(x) \\
&\leq C \int_{B_{3/2}} I_v(x, 1) \mathrm{d}\mu_l(x) < +\infty,
\end{aligned} \tag{7.5}$$

where we used: Fubini's Theorem and $1 + \lambda^{q_0} \leq {}^3/_2$ in the second inequality, $18\lambda < 1$ in the third, and $\lambda^{q_0-1} \leq 1$ and $\mu(B_{3/2}) < +\infty$ (by Theorem 1.1) in the last line. The conclusion now follows straightforwardly: indeed, by (7.5) we have that

$$\sum_{q\in\mathbb{N}} \beta_\mu^2(y, \lambda^q) < +\infty \quad \text{for } \mu_l\text{-a.e. } y \in B_1.$$

In view of (7.2), we can then apply Theorem 7.1 to conclude that E_l is $\mathcal{H}^{n-1}$-rectifiable. □

Remark 7.4. The rectifiability of the free boundary could also be deduced by following the argument by Naber and Valtorta, along the proof of the covering argument and the discrete Reifenberg Theorem: we refer to [31, 32] for more details.

8. Classification of blow-ups $\mathcal{H}^{n-1}$-a.e.

In this section we give the proof of the last main result of the paper, namely Theorem 1.3. We recall the rescalings for the blow-up procedure:

$$u_{x_0,r}(y) := \frac{r^{\frac{n+a}{2}}\, u(ry + x_0)}{H^{1/2}(x_0, r)} \quad \forall\, r < 1 - |x_0|,\ \forall\, y \in B_1. \tag{8.1}$$

In view of Remark 2.14, the functions $\bar{u}_{x_0,r}$ and $u_{x_0,r}$ have limits which differ only by a multiplicative constant. Therefore, Theorem 1.3 is proven once we show the same conclusions for the new rescalings (8.1).

8.1. **Stratification of the free boundary.** We start off with the first part of Theorem 1.3 regarding the estimate on the dimension of the set of points with frequency $\lambda \in \{2m, 2m-1+s, 2m+2s\}_{m\in\mathbb{N}\setminus\{0\}}$. We use a stratification argument for the nodal set $\mathcal{N}(u)$ of a solution u to the lower dimensional obstacle problem (1.1). This argument goes back to the work of Almgren [1, § 2.26]; here for convenience we follow [19].

We start recalling the definition of nodal points:

$$\mathcal{N}(u) := \Big\{(x',0) \in B'_R \,:\, u(x',0) = |\nabla_\tau u(x',0)| = \lim_{t\downarrow 0^+} t^a \partial_{n+1} u(x',t) = 0\Big\}.$$

Next we specify the main ingredients of [19, § 3.1] for the thin obstacle problem:

(a) the upper semi-continuous function $f : B'_1 \to \mathbb{R}$ given by

$$f(x) := \begin{cases} I_u(x,0^+) & \text{if } x \in \mathcal{N}(u), \\ 0 & \text{if } x \notin \mathcal{N}(u). \end{cases}$$

(b) the compact class of conical functions $\mathcal{G}(x) \subset L^\infty(\mathbb{R}^n)$, for every $x \in B'_1$, defined by

$$\mathcal{G}(x) := \begin{cases} \big\{I_w(\cdot,0^+) \,:\, w \in \mathrm{BU}(x)\big\} & \text{if } x \in \mathcal{N}(u), \\ \{0\} & \text{if } x \notin \mathcal{N}(u), \end{cases}$$

recall that $\mathrm{BU}(x)$ denotes the set of all blow-ups of u at x.

We need to verify that $\mathcal{G}(x)$ is a class of *compact conical* functions according to [19, Definition 3.3] (the arguments are analogous to those in [19, § 5.2], we repeat them for readers' convenience).

(1) An upper semi-continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *conical* if $g(z) = g(0)$ implies that

$$g(z+\lambda x) = g(z+x) \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, \lambda > 0.$$

Then, both the zero function and the frequency of homogeneous solutions w are conical by Lemma 5.2.

(2) A class $\mathcal{G}$ of conical functions is called *compact* if for every sequence $(g_j)_{j\in\mathbb{N}} \subset \mathcal{G}$ there exist a subsequence $(g_{j_i})_{i\in\mathbb{N}} \subset (g_j)_{j\in\mathbb{N}}$ and $g \in \mathcal{G}$ such that

$$\limsup_{i\to+\infty} g_{j_i}(y_i) \le g(y) \qquad \forall\, y \in \mathbb{R}^n,\ \forall\, (y_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n \text{ with } y_i \to y. \tag{8.2}$$

According to item (b), if $(g_j)_{j\in\mathbb{N}} \subset \mathcal{G}(x)$ we may assume without loss of generality g_j not identically 0 for j big. Then, $g_j = I_{w_j}(\cdot,0^+)$ and by Lemma 3.1 and Corollary 2.10 there exists a subsequence w_{j_i} converging to a homogeneous solution w (recall that $I_{w_j}(1) = I_u(x,0^+)$ and $H_{w_j}(1) = 1$). By a diagonal argument we have that $w \in \mathrm{BU}(x)$, and (8.2) follows from

$$\limsup_{i\to+\infty} I_{w_{j_i}}(y_i,0^+) \le \inf_{s>0} \limsup_{i\to+\infty} I_{w_{j_i}}(y_i,s) = \inf_{s>0} I_w(y,s) = I_w(y,0^+).$$

We discuss next the structural hypotheses [19, (i) - (ii) § 3.1]:

(i) $g(0) = f(x)$ for all $g \in \mathcal{G}(x)$, because $I_w(0^+) = I_u(x,0^+)$ for every blowup $w \in \mathrm{BU}(x)$;
(ii) for all $r_j \downarrow 0$ there exists a subsequence $(r_{j_i})_{i\in\mathbb{N}} \subset (r_j)_{j\in\mathbb{N}}$ and $w \in \mathcal{G}(x)$ such that $u_{x,r_{j_i}} \to w$; hence, for every $y \in \mathbb{R}^n$ and for every sequence $y_i \to y$, we have

$$\begin{aligned}\limsup_{i\to+\infty} I_u(x+r_{j_i}y_j,0^+) &\le \inf_{s>0} \limsup_{i\to+\infty} I_u(x+r_{j_i}y_j, r_{j_i}\, s) \\ &= \inf_{s>0} \limsup_{i\to+\infty} I_{u_{x,r_{j_i}}}(y_j,s) = \inf_{s>0} I_w(y,s) = I_w(y,0^+).\end{aligned}$$

We are then in the position to apply [19, Theorem 3.4] and conclude that the points whose blow-ups have spines with dimension not exceeding $l \in \{0,\ldots,n\}$ constitute a set of Hausdorff dimension at most l.

Theorem 8.1. *Let u be a solution of the thin obstacle problem (1.1) in B_R. For $l \in \{0,\ldots,n\}$, set $\Sigma_l(u) := \{x \in \mathcal{N}(u) : \dim S(w) \le l,\ \forall\, w \in \mathrm{BU}(x)\}$. Then, $\Sigma_0(u)$ is at most countable and $\dim_{\mathcal{H}} \Sigma_l(u) \le l$ for every $l \in \{1,\ldots,n\}$.*

The first assertion of Theorem 1.3 is now a direct consequence.

Proof of Theorem 1.3: part I. We first show that $\dim S(w) \leq n-1$ for every $w \in \mathrm{BU}(x)$ with $x \in \mathcal{N}(u)$. To this aim, we observe that by the definition of nodal set we have that $0 \in \mathcal{N}(w)$ for every $w \in \mathrm{BU}(x)$ with $x \in \mathcal{N}(u)$. On the other hand, using the notation in Theorem 8.1, $\Sigma_n(u) \setminus \Sigma_{n-1}(u) = \emptyset$ as noticed in Section 5.2. Indeed, the only non-trivial homogeneous solutions with n-dimensional spine are the functions $w_c := c\,|x_{n+1}|^{2s}$ with $c < 0$, and by direct computation $\mathcal{N}(w_c) = \emptyset$.

Therefore, for every $x \in \Gamma(u) \setminus \Sigma_{n-2}(u)$ there exists at least a blowup $w \in \mathrm{BU}(x)$ with an $(n-1)$-dimensional spine $S(w)$, *i.e.* with $w \in \mathcal{H}^{\mathrm{top}}$. Thus, by the classification of all homogeneous solutions with maximal spine in Lemma 5.3, the limiting frequency at any point $x \in \Gamma(u) \setminus \Sigma_{n-2}(u)$ satisfies

$$I_u(x, 0^+) = I_w(0, 0^+) \in \{2m, 2m-1+s, 2m+2s\}_{m \in \mathbb{N}\setminus\{0\}}.$$

Taking into consideration that $\dim_{\mathcal{H}} \Sigma_{n-2}(u) \leq n-2$ by Theorem 8.1, we conclude the proof. □

8.2. **Uniqueness of blow-ups with frequency $2m$ and $2m-1+s$.** For the second part of Theorem 1.3 we need an extension of the classification result in Lemma 5.3 to the λ-homogeneous (even symmetric with respect to x_{n+1}) solutions of

$$\begin{cases} \mathrm{div}(|x_{n+1}|^a \nabla u) = 0 & B_1 \setminus \Lambda(u) \\ u = 0 & \Lambda(u), \end{cases} \tag{8.3}$$

with $\lambda \in \{2m, 2m-1+s\}_{m \in \mathbb{N}\setminus\{0\}}$ and $\{x \cdot e = x_{n+1} = 0\} \subseteq \Lambda(u)$ for some unit vector $e \in \mathbb{R}^n \times \{0\}$. The main differences with Lemma 5.3 are that neither the unilateral obstacle condition nor any invariance assumption of the solutions (*i.e.* the assignment of the spine) are imposed in this framework. In the ensuing statement we keep the notation introduced in Lemma 5.3.

Proposition 8.2. *Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be a non-trivial λ-homogeneous weak solution of (8.3), even w.r.to x_{n+1}, such that $\lambda \in \{2m, 2m-1+s\}_{m \in \mathbb{N}\setminus\{0\}}$ and $\{x \cdot e = x_{n+1} = 0\} \subseteq \Lambda(u)$ for some unit vector $e \in \mathbb{R}^n \times \{0\}$. Then, there exists $c > 0$ such that $u(x) = c\,h_{2m}(x \cdot e, x_{n+1})$ or $u(x) = c\,h_{2m-1+s}(x \cdot e, x_{n+1})$ or $u(x) = c\,h_{2m-1+s}(-x \cdot e, x_{n+1})$.*

The proof is postponed to Proposition A.3 in the appendix. Given it for granted, we proceed with the conclusion of the proof of Theorem 1.3.

Proof of Theorem 1.3: part II. By Theorem 1.2 there exist at most countably many C^1-regular submanifolds $\{M_i\}_{i \in \mathbb{N}}$ such that $\mathcal{H}^{n-1}(\Gamma(u) \setminus \cup_{i \in \mathbb{N}} M_i) = 0$. We consider the sets $\Gamma_i(u) := \Gamma(u) \cap M_i$ and

$$\Gamma_i'(u) := \left\{ x \in \Gamma_i(u) \,:\, \lim_{r \downarrow 0^+} \frac{\mathcal{H}^{n-1}(B_r(x_0) \cap \Gamma_i(u))}{\omega_{n-1}\, r^{n-1}} = 1 \right\}.$$

Note that $\mathcal{H}^{n-1}(\Gamma_i(u) \setminus \Gamma_i'(u)) = 0$ for every $i \in \mathbb{N}$ by Besicovitch's differentiation theorem (cp. [2, Theorem 2.22]). We show that for every $i \in \mathbb{N}$ and for every $x_0 \in \Gamma_i'(u)$ the conclusion of Theorem 1.3 holds, namely if $I(x_0, 0^+) = \lambda \in \{2m, 2m-1+s\}_{m \in \mathbb{N}\setminus\{0\}}$, then there exists a unit vector e_{x_0} with $e_{x_0} \perp \mathrm{Tan}_{x_0} M_i$ at x_0 such that

$$u_{x_0,r} \to h_\lambda(x \cdot e_{x_0}, x_{n+1}) \quad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, |x_{n+1}|^a \mathcal{L}^{n+1}) \text{ as } r \downarrow 0, \tag{8.4}$$

where h_λ are the functions in Lemma 5.3, and $\mathrm{Tan}_{x_0} M_i$ is the linear tangent space to M_i at x_0: *i.e.*

$$v \in \mathrm{Tan}_{x_0} M_i \quad \Longleftrightarrow \quad \exists\, (x_l)_{l \in \mathbb{N}} \subset M_i \text{ such that } \lim_{l \to +\infty} \frac{x_l - x_0}{|x_l - x_0|} = v.$$

To this aim we consider the compact sets

$$K_r := \left\{ y \in \overline{B}_1 \,:\, x_0 + r\,y \in \Gamma_i(u) \right\} \quad \text{for} \quad r \leq r_0 := \frac{1}{2}\,(1 - |x_0|).$$

By Blaschke compactness theorem (cp. [2, Theorem 6.1]) the sequence of sets $\{K_r\}_{r \in (0, r_0]}$ is pre-compact in the Hausdorff distance on $\overline{B}_1$: namely, given any sequence $(0, r_0] \ni r_i \to 0$, there exists a subsequence $(r_{i_k})_{k \in \mathbb{N}}$ and a compact set $K_0 \subseteq \overline{B}_1$ such that we have $\lim_k \mathrm{dist}_{\mathcal{H}}(K_{r_{i_k}}, K_0) = 0$, *i.e.*

(a) any point $x \in K_0$ is an accumulation point for a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in K_{r_{i_k}}$;
(b) if $x_k \in K_{r_{i_k}}$, then any accumulation point of $(x_k)_{k\in\mathbb{N}}$ belongs to K_0.

We proceed now with the proof of (8.4) in three steps.

1. Let $r_j \downarrow 0$ be such that $\mathrm{dist}_{\mathcal{H}}(K_{r_j}, K_0) \to 0$ for some compact set K_0. Then

$$\mathrm{Tan}_{x_0} M_i \cap \overline{B}_1 \subseteq K_0.$$

Assuming this is not the case, there exists an open ball $B_\rho(y_0) \subset B_1 \subset \mathbb{R}^n$ with $y_0 \in \mathrm{Tan}_{x_0} M_i$ such that $\big(\Gamma_i(u) - x_0\big)/r_j \cap B_\rho(y_0) = \emptyset$. In particular, for sufficiently large j we have that

$$\begin{aligned}\mathcal{H}^{n-1}\big({}^{(\Gamma_i(u)-x_0)}/{}_{r_j} \cap \overline{B}_1\big) &= \mathcal{H}^{n-1}\big({}^{(\Gamma_i(u)-x_0)}/{}_{r_j} \cap \overline{B}_1 \setminus B_\rho(y_0)\big)\\ &\le \mathcal{H}^{n-1}\big({}^{(M_i-x_0)}/{}_{r_j} \cap \overline{B}_1 \setminus B_\rho(y_0)\big) \le \big(1+o(1)\big)\,\omega_{n-1}\left(1-\rho^{n-1}\right) \quad \text{as } j \to +\infty,\end{aligned}$$

since $\mathcal{H}^{n-1}\big({}^{(M_i-x_0)}/{}_{r_j} \cap A\big) \to \mathcal{H}^{n-1}\big(\mathrm{Tan}_{x_0} M_i \cap A\big)$ for every open set $A \subset \mathbb{R}^n$. This yields

$$\limsup_{j\to+\infty} \frac{\mathcal{H}^{n-1}\big(\Gamma_i(u) \cap B_{r_j}(x_0)\big)}{\omega_{n-1} r_j^{n-1}} \le 1 - \rho^{n-1},$$

against the assumption $x_0 \in \Gamma_i'(u)$.

2. In particular, it follows that, if u_{x_0} is any blow-up limit of u at $x_0 \in \Gamma_i'(u)$, then

$$\mathrm{Tan}_{x_0} M_i \subseteq \big\{u_{x_0} = 0\big\}. \tag{8.5}$$

Indeed, set $Y_r := \big\{u_{x_0,r} = 0\big\} \cap \overline{B}_1$ and note that $K_r \subset Y_r$. If $Y_0 \subset \overline{B}_1$ is any Hausdorff limit of a sequence Y_{r_j}, then $Y_0 \subset \big\{u_{x_0} = 0\big\} \cap \overline{B}_1$, because $u_{x_0,r_j}(z_j) \to u_{x_0}(z_0)$ for every $z_j \in Y_{r_j}$ with $z_j \to z_0$ (thanks to the uniform convergence of $u_{x_0,r_{i_k}}$). In particular, being $K_{r_j} \subset Y_{r_j}$, the conclusion follows from step 1 and the homogeneity of u_{x_0}.

3. We now conclude the proof of (8.4). Assume without loss of generality that $\mathrm{Tan}_{x_0} M_i = \{x_n = x_{n+1} = 0\}$. By Proposition 2.11 we have that $\mathrm{BU}(x_0) \subseteq \mathcal{H}_\lambda$ with $\lambda = I_u(x_0, 0^+)$, and we distinguish two possibilities (recall also that the blow-ups are renormalized so to have $H_{u_{x_0}}(1) = 1$):

(1) $I_u(x_0, 0^+) = 2m$. By Proposition 8.2 the blow-up u_{x_0} needs to be $h_{2m}(x_n, x_{n+1})$, because this function is the only blow-up with frequency $2m$ and contact set containing $\mathrm{Tan}_{x_0} M_i = \{x_n = x_{n+1} = 0\}$ by (8.5);
(2) $I_u(x_0, 0^+) = 2m - 1 + 2s$. By Proposition 8.2 every blow-up u_{x_0} is given by either $h^+ = h_{2m-1+s}(x_n, x_{n+1})$ or $h^- = h_{2m-1+s}(-x_n, x_{n+1})$.

In order to infer the uniqueness of the blowup in this last case, we exploit the connectedness of the set of blow-up limits. Namely, assume that there exist $r_i \downarrow 0$ and $\rho_i \downarrow 0$ such that $u_{x_0,r_i} \to h^+$ and $u_{x_0,\rho_i} \to h^-$; up to passing to subsequences, we may take $r_i < \rho_i < r_{i+1}$. Then, by continuity there exists $t_i \in (r_i, \rho_i)$ such that

$$\|u_{x_0,t_i} - h^+\|_{L^2(B_1)} = \|u_{x_0,t_i} - h^-\|_{L^2(B_1)}.$$

Since the sequence $(u_{x_0,t_i})_{i\in\mathbb{N}}$ has no subsequence converging either to h^+ or to h^-, this gives a contradiction and concludes the proof of Theorem 1.3. □

8.3. **Concerning the optimality of Theorem 1.3.** For every $e \in \mathbb{R}^{n+1}$ with $|e| = 1$ and $e \cdot e_{n+1} = 0$, the functions $u(x) = h_{2m}(x \cdot e, x_{n+1})$ and $u = h_{2m-1+s}(x \cdot e, x_{n+1})$ are examples of solutions to the lower dimensional problem (1.1) in any ball B_R whose free boundary $\Gamma(u)$ is $(n-1)$-dimensional and is made of points with frequency $2m$ and $2m - 1 + s$, respectively. Note that the latters are explicit cases in which $\Gamma(u) = \mathrm{Other}(u)$.

On the other hand, as pointed out in the introduction, at the best of our knowledge there are no explicit examples of solutions to the lower dimensional obstacle problem (1.1) with free boundary points with frequency $2m + 2s$ with $m \in \mathbb{N} \setminus \{0\}$ (note that, although $h_{2m+2s}(x \cdot e, x_{n+1})$ are solutions, $\Gamma(h_{2m+2s}) = \emptyset$).

Such points do not occur in the one dimensional case $n = 1$. Following the argument of [23, Remark 1.2.8] for $s = 1/2$, assume that $0 \in \Gamma(u)$ is a point with frequency $2m + 2s$. Then, one can find a sequence $(t_k, 0) \in \mathbb{R}^2$ with $\lim_k t_k = 0$ such that $u(t_k, 0) > 0$ and, therefore, from (2.4),

$$\lim_{x_2 \downarrow 0} x_2^a \partial_{x_2} u(t_k, x_2) = 0. \tag{8.6}$$

Taking the rescalings $u_{0,t_k/2}$, up to passing to a subsequence (not relabeled) there exists a blowup $w \in \mathrm{BU}(0)$ such that (cp. (2.20)):

$$\operatorname{sign}(x_2)|x_2|^a \partial_{x_2} u_{0,t_k/2} \to \operatorname{sign}(x_2)|x_2|^a \partial_{x_2} w \quad \text{in } C^0_{\mathrm{loc}}(B_1).$$

Note that necessarily $w = h_{2m+2s}$, because there exists a unique blowup with frequency $2m + 2s$. Moreover, from (8.6) we have that $\lim_{x_2 \downarrow 0} x_2^a \partial_{x_2} w(1/2, x_2) = 0$. On the contrary a direct computation shows that $\lim_{x_2 \downarrow 0} x_2^a \partial_{x_2} h_{2m+2s}(1/2, x_2) < 0$, thus leading to a contradiction and implying that there cannot exist free boundary points with frequency $2m + 2s$ for $n = 1$.

Potential points with frequency $2m+2s$ are sometimes referred to in the literature as *degenerate points* (see the final section of [23]). It is a tempting conjecture to claim that there are actually none. If this were the case, Theorem 1.3 would then be optimal, both concerning the uniqueness of blow-ups at $\mathcal{H}^{n-1}$-almost all points of the free boundary, and the classification of the frequency at $\mathcal{H}^{n-2}$-almost all points of the free boundary.

Appendix A. Homogeneous solutions

In this appendix we collect some results concerning homogeneous solutions to the thin obstacle problem and more generally to the corresponding system of Euler–Lagrange equations. Therefore, we consider functions $u : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$u(x) = |x|^\lambda u\left(x/|x|\right) \quad \forall\, x \neq 0$$

for some $\lambda \geq 1 + s$ (this restriction being in accordance with the homogeneity of all possible blow-ups).

A.1. **Two-dimensional homogeneous solutions.** Here we provide a classification of the homogeneous solutions to the equation

$$\begin{cases} \operatorname{div}(|x_2|^a \nabla u) = 0 & B_1 \setminus \Lambda(u), \\ u = 0 & \Lambda(u), \end{cases} \tag{A.1}$$

in the two dimensional case, *i.e.* for $n = 1$. Thus, necessarily, the contact set is a cone, and we have:

(i) $\Lambda(u) = \{x_1 = x_2 = 0\}$,
(ii) $\Lambda(u) = \{x_1 \leq 0,\, x_2 = 0\}$ or $\Lambda(u) = \{x_1 \geq 0,\, x_2 = 0\}$,
(iii) $\Lambda(u) = \{x_2 = 0\}$.

Correspondingly, we introduce three classes of functions Φ_m, Ψ_m and Π_m for $m \in \mathbb{N}$, that are explicitly defined as follows:

$$\Phi_m(x_1, x_2) := \sum_{k=0}^{\lfloor m/2 \rfloor} \alpha_k\, x_1^{m-2k} x_2^{2k}, \tag{A.2}$$

$$\Psi_m(x_1, x_2) := \left(\sqrt{x_1^2 + x_2^2} + x_1\right)^s \sum_{k=0}^{m} \beta_k \left(\sqrt{x_1^2 + x_2^2} - x_1\right)^k \left(\sqrt{x_1^2 + x_2^2}\right)^{m-k}, \tag{A.3}$$

$$\Pi_m(x_1, x_2) := |x_2|^{2s} \sum_{k=0}^{\lfloor m/2 \rfloor} \gamma_k x_2^{2k} \frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}} p(x_1), \tag{A.4}$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number, and

$$\alpha_0 := 1, \quad \alpha_{k+1} := -\frac{(m-2k)(m-2k-1)}{4(k+1)(k+1-s)}\, \alpha_k \qquad k \in \{0, \ldots, \lfloor m/2 \rfloor - 1\},$$

$$\beta_k := \frac{(m+1)_k(-m)_k}{2^k k!\,(1-s)_k} \qquad k \in \{0,\dots,m\},$$

$$\gamma_k := \frac{(-1)^k}{4^k\, k!\,(1+s)_k} \qquad k \in \{0,\dots,\lfloor m/2 \rfloor\}, \quad p \text{ is a } m\text{-homogenous polynomial},$$

and the (increasing) Pochhammer symbol is defined by

$$(q)_l := \begin{cases} 1 & \text{if } l = 0,\\ q\,(q+1)\cdots(q+l-1) & \text{if } l \in \mathbb{N}\setminus\{0\}. \end{cases}$$

We establish the ensuing classification result (for related issues see [12, 34]).

Proposition A.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be λ-homogeneous, even symmetric w.r.to x_2, and assume that u is a weak solution of (A.1). Then, one of the following occurs:*

(i) $\Lambda(u) = \{x_1 = x_2 = 0\}$, $\lambda = m \in \mathbb{N}\setminus\{0\}$ *and u is a multiple of Φ_m;*
(ii) $\Lambda(u) = \{x_1 \le 0,\, x_2 = 0\}$ *(resp.* $\Lambda(u) = \{x_1 \ge 0,\, x_2 = 0\}$*), $\lambda = m + s$ for some $m \in \mathbb{N}$ and u is a multiple of Ψ_m (resp. of $\Psi_m(-x_1, x_2)$);*
(iii) $\Lambda(u) = \{x_2 = 0\}$, *$\lambda = m + 2s$ for some $m \in \mathbb{N}$ and u is a multiple of Π_m.*

Moreover, if u is a solution to the lower dimensional obstacle problem, then m is even in (i) and (iii), and m is odd in (ii).

For the proof we need to introduce the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;\cdot) : \mathbb{C} \to \mathbb{C}$ defined by

$$ {}_2F_1(\alpha,\beta;\gamma;z) := \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\,\frac{z^k}{k!}, \tag{A.5}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, γ not a negative integer. The power series defining ${}_2F_1$ is converging for $|z| < 1$, and it can be analytically continued elsewhere. In what follows we shall use several properties of ${}_2F_1$ for which we refer to the Digital Library of Mathematical Functions, always quoting the precise formulas employed in the derivation and referring to their enumeration in [33].

We warn the reader that, with a slight abuse of notation, in this section Γ shall denote both the free boundary of a solution and the Euler's Gamma-function on the complex plane, extended to $Re(z) \le 0$ by analytic continuation using the identity $\Gamma(1+z) = z\Gamma(z)$. In particular, Γ turns out to be a meromorphic function with no zeros and simple poles at $z = -m$, $m \in \mathbb{N}$. Thus, we adopt the convention that $\Gamma^{-1}(-m) = 0$ for all $m \in \mathbb{N}$.

Proof of Proposition A.1. Using polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ with $r > 0$ and $\theta \in [0,\pi]$, let $v(r,\theta) := u\big(r\cos\theta, r\sin\theta\big) = r^\lambda\, y(\theta)$. The Euler-Lagrange equation (A.1) then reads as

$$\mathcal{L}_{a,\lambda}\big[y(\theta)\big] := y''(\theta) + a\,\mathrm{cotg}\,\theta\, y'(\theta) + \lambda\,(\lambda + a)\, y(\theta) = 0 \qquad \theta \in (0,\pi), \tag{A.6}$$

with boundary conditions:

- case (i)
$$\lim_{\theta\downarrow 0^+} (\sin\theta)^{1-2s} y'(\theta) = 0 \quad \text{and} \quad \lim_{\theta\uparrow \pi^-} (\sin\theta)^{1-2s} y'(\theta) = 0,$$
- case (ii) (by symmetry we assume $\Lambda(u) = \{x_1 \le 0, x_2 = 0\}$)
$$\lim_{\theta\downarrow 0^+} (\sin\theta)^{1-2s} y'(\theta) = 0 \quad \text{and} \quad y(\pi) = 0,$$
- case (iii)
$$y(0) = 0 \quad \text{and} \quad y(\pi) = 0.$$

The change of variable $y(\theta) = (\sin\theta)^s\, h(\cos\theta)$ transforms the ODE for y in (A.6) into an associated Legendre differential equation for h. More precisely, we get for $\nu = \lambda - s$ and $\nu \ge 1$

$$(1-x^2)h''(x) - 2xh'(x) + \left(\nu^2 + \nu - \frac{s^2}{1-x^2}\right) h(x) = 0 \qquad x \in (-1,1), \tag{A.7}$$

with the following boundary conditions:

- case (i)

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\big(sxh(x)-(1-x^2)h'(x)\big)=0, \tag{A.8}$$

$$\lim_{x\downarrow -1}(1-x^2)^{-s/2}\big(sxh(x)-(1-x^2)h'(x)\big)=0, \tag{A.9}$$

- case (ii)

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\big(sxh(x)-(1-x^2)h'(x)\big)=0, \tag{A.10}$$

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}h(x)=0, \tag{A.11}$$

- case (iii)

$$\lim_{x\uparrow 1}(1-x^2)^{s/2}h(x)=0, \tag{A.12}$$

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}h(x)=0. \tag{A.13}$$

The associated Legendre equation can be solved explicitly in terms of the hypergeometric function ${}_2F_1$ (cf. (A.5)). A generic solution in the interval $(-1,1)$ is given by

$$h(x)=A_1\,P^s_\nu(x)+A_2\,P^{-s}_\nu(x)$$

where A_1, $A_2\in\mathbb{R}$ and

$$P^{\pm s}_\nu(x):=\frac{1}{\Gamma(1\mp s)}\left(\frac{1+x}{1-x}\right)^{\pm s/2}{}_2F_1\Big(\nu+1,-\nu,1\mp s,\frac{1-x}{2}\Big), \tag{A.14}$$

$$=\frac{1}{\Gamma(1\mp s)}\big(1-x^2\big)^{\mp s/2}{}_2F_1\Big(\mp s-\nu,1\mp s+\nu,1\mp s,\frac{1-x}{2}\Big), \tag{A.15}$$

(cf. [33, (14.3.1), (14.3.2), Section 15.1 and (15.8.1)]).

1. Dirichlet boundary conditions. We now proceed computing the boundary conditions in terms of the explicit representations (A.14) and (A.15). First, note that by continuity of ${}_2F_1(\alpha,\beta,\gamma,\cdot)$ and since ${}_2F_1(\alpha,\beta,\gamma,0)=1$ for all α, β and γ, we get

$$\lim_{x\uparrow 1}(1-x^2)^{s/2}P^s_\nu(x)=\frac{2^s}{\Gamma(1-s)}\quad\text{and}\quad\lim_{x\uparrow 1}(1-x^2)^{s/2}P^{-s}_\nu(x)=0,$$

from which we get

$$\lim_{x\uparrow 1}(1-x^2)^{s/2}h(x)=A_1\,\frac{2^s}{\Gamma(1-s)}. \tag{A.16}$$

For the corresponding limit values as $x\downarrow -1$ we use [33, (15.4.20)] to infer

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}P^{-s}_\nu(x)=\frac{2^s\,\Gamma(s)}{\Gamma(s-\nu)\,\Gamma(1+s+\nu)},$$

and from (A.15) and [33, (15.4.20)]

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}P^{s}_\nu(x)=\frac{2^s\,\Gamma(s)}{\Gamma(-\nu)\,\Gamma(1+\nu)},$$

from which

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}h(x)=A_1\,\frac{2^s\,\Gamma(s)}{\Gamma(-\nu)\,\Gamma(1+\nu)}+A_2\,\frac{2^s\,\Gamma(s)}{\Gamma(s-\nu)\,\Gamma(1+s+\nu)}. \tag{A.17}$$

2. Neumann boundary conditions. For what concerns the boundary conditions involving the derivative of h we use [33, (15.5.1)] to compute

$$\frac{d}{dx}P^{\pm s}_\nu(x)=\frac{1}{\Gamma(1\mp s)}\Big[\mp s\frac{(1\pm x)^{s/2-1}}{(1\mp x)^{s/2+1}}{}_2F_1\big(\nu+1,-\nu,1\mp s,\frac{1-x}{2}\big)$$
$$+\frac{\nu(\nu+1)}{2(1\mp s)}\left(\frac{1+x}{1-x}\right)^{\pm s/2}{}_2F_1\big(\nu+2,1-\nu,2\mp s,\frac{1-x}{2}\big)\Big].$$

Hence, we get

$$(1-x^2)^{-s/2}\left(sxP_\nu^{\pm s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{\pm s}(x)\right)=\mp\frac{(1\mp x)^{1-s}}{\Gamma(1\mp s)}\cdot$$
$$\cdot\Big[s\cdot{}_2F_1\big(\nu+1,-\nu,1\mp s,\frac{1-x}{2}\big)\pm\frac{\nu(\nu+1)}{2(1+s)}(1\pm x)\cdot{}_2F_1\big(\nu+2,1-\nu,2\mp s,\frac{1-x}{2}\big)\Big].\qquad(A.18)$$

From the latter formula we immediately conclude that

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\left(sxP_\nu^{s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{s}(x)\right)=0,$$

and

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\left(sxP_\nu^{-s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{-s}(x)\right)=\frac{s\,2^{1-s}}{\Gamma(1+s)}.$$

Therefore, we have

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\big(sxh(x)-(1-x^2)h'(x)\big)=A_2\,\frac{s\,2^{1-s}}{\Gamma(1+s)}.\qquad(A.19)$$

In addition, from (A.18), from the linear transformation of variable rule for ${}_2F_1$ in [33, (15.8.4)] and from [33, Section 15.5], elementary calculations lead to

$$(1-x^2)^{-s/2}\left(sxP_\nu^{s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{s}(x)\right)=\frac{\pi(1-x)^{1-s}}{\sin(s\pi)\Gamma(-s-\nu)\,\Gamma(1-s+\nu)\,\Gamma(1+s)}\cdot$$
$$\cdot\left(s\cdot{}_2F_1\Big(\nu+1,-\nu,1+s,\frac{1+x}{2}\Big)-\frac{\nu(\nu+1)}{2}(1+x)\cdot{}_2F_1\Big(\nu+2,1-\nu,1+s,\frac{1+x}{2}\Big)\right)$$
$$+\frac{2^{s-1}\pi(s+\nu)(1-s+\nu)}{\sin(s\pi)\Gamma(\nu+1)\,\Gamma(-\nu)\,\Gamma(2-s)}(1-x^2)^{1-s}\cdot{}_2F_1\Big(1-s-\nu,2-s+\nu,2-s,\frac{1+x}{2}\Big).$$

In turn, this implies

$$\lim_{x\downarrow -1}(1-x^2)^{-s/2}\left(sxP_\nu^{s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{s}(x)\right)=\frac{2^{1-s}\pi}{\sin(s\pi)\Gamma(-s-\nu)\,\Gamma(1-s+\nu)\,\Gamma(s)}.$$

Finally, by [33, (15.8.1)] we rewrite (A.18) for P_ν^{-s} as

$$(1-x^2)^{-s/2}\Big(sxP_\nu^{-s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{-s}(x)\Big)$$
$$=\frac{1}{\Gamma(1+s)}\cdot\Big[s(1+x)^{1-s}{}_2F_1\big(\nu+1,-\nu,1+s,\frac{1-x}{2}\big)$$
$$-\frac{\nu(\nu+1)}{2^s(1+s)}(1-x)\cdot{}_2F_1\big(s-\nu,1+s+\nu,2+s,\frac{1-x}{2}\big)\Big],$$

and infer from [33, (15.4.20)]

$$\lim_{x\downarrow -1}(1-x^2)^{-s/2}\left(sxP_\nu^{-s}(x)-(1-x^2)\frac{d}{dx}P_\nu^{-s}(x)\right)=-\frac{\nu(\nu+1)}{2^{s-1}}\frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(2+\nu)\Gamma(1-\nu)},$$

i.e.

$$\lim_{x\downarrow -1}(1-x^2)^{-s/2}\big(sxh(x)-(1-x^2)h'(x)\big)=A_1\,\frac{2^{1-s}\pi}{\sin(s\pi)\Gamma(-s-\nu)\,\Gamma(1-s+\nu)\,\Gamma(s)}$$
$$-A_2\,\frac{\nu(\nu+1)}{2^{s-1}}\frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(2+\nu)\Gamma(1-\nu)}\qquad(A.20)$$

3. By means of (A.16), (A.17), (A.19) and (A.20) we are able to complete the classification by discussing all the cases (i) – (iii). We start off with case (i): using (A.19) and (A.20) we deduce that $A_2=0$ and $\nu+s=m\in\mathbb{N}$ (in order to have $1/\Gamma(-s-\nu)=0$). Therefore,

$$h(x)=A_1P_\nu^{s}(x)\overset{(A.15)}{=}\frac{2^s\,A_1}{\Gamma(1-s)}(1-x^2)^{-s/2}\,{}_2F_1\Big(1+m-2s,-m,1-s,\frac{1-x}{2}\Big).$$

In particular, $(1-x^2)^{s/2}h(x)$ is a polynomial of degree $d \le m$ (or a constant if $m = 0$), as $(-m)_k = 0$ for every $k \ge m+1$. The case $m = 0$ implies y to be constant and thus $u \equiv 0$, which is excluded from the condition $\Lambda(u) = \{x_1 = x_2 = 0\}$. Hence, $m > 0$ and y is a polynomial of degree d in $\cos\theta$. As for every $k \ge 0$

$$\mathcal{L}_{\lambda,a}\big[\,(\cos\theta)^k\big] = \big(\lambda(\lambda+a) - k(k+a)\big)\,(\cos\theta)^k + k(k-1)\,(\cos\theta)^{k-2},$$

we infer that $d = \lambda = \nu + s = m$ and that y depends only on powers of $\cos\theta$ with the same parity as m:

$$y(\cos\theta) = a_m\,(\cos\theta)^m + a_{m-2}\,(\cos\theta)^{m-2} + \cdots + a_{m-\lfloor m/2\rfloor}\,(\cos\theta)^{m-\lfloor m/2\rfloor}, \quad a_m \ne 0.$$

Therefore, u is an m-homogeneous polynomial of the form in (A.2) and by a direct computation

$$\operatorname{div}\big(x_2^a\nabla u(x)\big) = \sum_{k=0}^{m-1}\Big((m-2k)(m-2k-1)\,\alpha_k + 2(k+1)(2k+1+a)\,\alpha_{k+1}\Big)\,x_1^{m-2k-2}x_2^{2k+a} = 0$$

we conclude the explicit form of the coefficients α_k.

Next we discuss case (ii): from (A.19) we get $A_2 = 0$ and from (A.17) we get $\nu \in \mathbb{N}$. Thus ${}_2F_1(\nu+1, -\nu, 1-s, \cdot)$ is a polynomial of degree at most $\nu = m$ with $m \in \mathbb{N}\setminus\{0\}$. The corresponding representation formula in (A.3) follows at once from

$$\begin{aligned} u\big(r\cos\theta, r\sin\theta\big) &= A_1\,r^{m+s}\,(\sin\theta)^s\,P_\nu^s(\cos\theta) \\ &\overset{(A.14)}{=} A_1\,r^{m+s}\sum_{k=0}^{m}\frac{(m+1)_k(-m)_k}{2^k k!\,(1-s)_k}\big(1-\cos\theta\big)^k\big(1+\cos\theta\big)^s. \end{aligned}$$

We discuss case (iii): from (A.16) and (A.17) we get that $A_1 = 0$ and $\nu - s = m \in \mathbb{N}$ and the representation formula for solutions in (A.4) follows by direct verification (alternatively one can also derive it from the explicit formula in terms of the hypergeometric function).

4. Finally, we discuss the case of solutions u to the lower dimensional obstacle problem (1.1). In particular, u solves (A.1), $u|_{B_1'} \ge 0$ and the normal weighted derivative satisfies a sign condition. Thus, the following additional boundary conditions need to be satisfied by y:

$$\lim_{\theta\downarrow 0^+}(\sin\theta)^{1-2s}y'(\theta) \le 0 \quad \text{and} \quad \lim_{\theta\uparrow\pi^-}(\sin\theta)^{1-2s}y'(\theta) \ge 0,$$
$$y(0) \ge 0 \quad \text{and} \quad y(\pi) \ge 0.$$

In turn, these for the function h translate into

$$\lim_{x\uparrow 1}(1-x^2)^{-s/2}\big(sxh(x) - (1-x^2)h'(x)\big) \le 0, \tag{A.21}$$

$$\lim_{x\downarrow -1}(1-x^2)^{-s/2}\big(sxh(x) - (1-x^2)h'(x)\big) \ge 0, \tag{A.22}$$

$$\lim_{x\uparrow 1}(1-x^2)^{s/2}h(x) \le 0, \tag{A.23}$$

$$\lim_{x\downarrow -1}(1-x^2)^{s/2}h(x) \ge 0. \tag{A.24}$$

We can then discuss the implications of (A.21) – (A.24) for the tree cases (i) – (iii). In case (i), by (A.16) and (A.23) we get $A_1 \ge 0$; similarly, by (A.17) and (A.24) we get that $\Gamma(-\nu) > 0$, *i.e.* $2m-1 \le \nu \le 2m$ for some $m \in \mathbb{N}\setminus\{0\}$. Since $\nu + s \in \mathbb{N}$, we conclude that $\nu + s = 2m$.

In case (ii), using (A.16) and (A.23) we conclude that $A_1 \ge 0$; moreover, from (A.20) and (A.22) we infer that $\Gamma(-s-\nu) \ge 0$ and therefore $2m+1 \le \nu + s \le 2m+2$ for some $m \in \mathbb{N}$. In particular, since $\nu \in \mathbb{N}$, we have that $\nu = 2m+1$ is odd.

Finally, in case (iii), using (A.19) in (A.21) and (A.20) in (A.22) we deduce that $A_2 \ge 0$ and $\Gamma(1-\nu) \le 0$, from which it follows that $\nu - s = 2m$ is even. □

Using Proposition A.1 we now complete the proof of the classification of global solutions $u \in \mathcal{H}^{top}$ with $(n-1)$-dimensional spine in Lemma 5.3.

Proof of Lemma 5.3. For every $u \in \mathcal{H}^{top}$, we have that u depends on x_{n+1} and only one in-plane variable, *i.e.* $u(x) = v(x \cdot e, x_{n+1})$ for some $v : \mathbb{R}^2 \to \mathbb{R}$ and for some unit vector $e \in \mathbb{R}^{n+1}$ with $e \cdot e_{n+1} = 0$. In particular, v is a two-dimensional solution to the lower dimensional obstacle problem in $\mathbb{R}^2$. Therefore, by Proposition A.1 we know that $\lambda \in \{2m, 2m-1+s, 2m+2s\}_{m\in\mathbb{N}\setminus\{0\}}$ and v is one of the functions in (A.2) – (A.4). The statements about $\Gamma(u)$, $\Lambda(u)$, $\mathcal{N}(u)$ and $S(u)$ follow from the explicit formulas therein. □

A.2. **Further classification results.** Here we provide a proof to Proposition 8.2. We split the argument in two parts. We start off classifying in any dimension all λ-homogeneous solutions (even symmetric with respect to x_{n+1}) of

$$\begin{cases} -\mathrm{div}(|x_{n+1}|^a \nabla u) = 0 & B_1 \setminus \Lambda(u) \\ u = 0 & \Lambda(u) \end{cases} \tag{A.25}$$

such that $\lambda \in [1+s, 2+s)$ and having as contact set $\Lambda(u)$ one of the following

(i) $\Lambda(u) = \{x_n = x_{n+1} = 0\}$,
(ii) $\Lambda(u) = \{x_n \le 0,\ x_{n+1} = 0\}$,
(iii) $\Lambda(u) = \{x_{n+1} = 0\}$.

We follow the arguments in [12, Lemma 5.3] and [24, Lemma A.3], in which the case $\lambda = 1+s$ with $\Lambda(u) = \{x_n \le 0,\ x_{n+1} = 0\}$ is addressed. To this aim we introduce the following notation: $x = (x'', x_n, x_{n+1}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$.

Lemma A.2. *Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be a λ-homogeneous solution of (A.25), even symmetric w.r.to x_{n+1}, with $\lambda \in [1+s, 2+s)$ and $\Lambda(u)$ one of the sets in (i) – (iii) above. Then, the following occurs:*

- *in case (i), $\lambda = 2$ and there exists a 1-homogeneous polynomial $q : \mathbb{R}^{n-1} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that*
$$u(x) = q(x'')\, x_n + c\, \Phi_2(x_n, x_{n+1}); \tag{A.26}$$
- *in case (ii), $\lambda = 1+s$ and there exists a 1-homogeneous polynomial $q : \mathbb{R}^{n-1} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that*
$$u(x) = q(x'') \left(x_n + \sqrt{x_n^2 + x_{n+1}^2} \right)^s + c\, \Psi_1(x_n, x_{n+1}); \tag{A.27}$$
- *in case (iii), $\lambda = 1+2s$ and there exists a 1-homogeneous polynomial $q : \mathbb{R}^n \to \mathbb{R}$ such that*
$$u(x) = |x_{n+1}|^{2s}\, q(x'). \tag{A.28}$$

Proof. **1.** In case (i), since $\mathcal{H}^n(\Lambda(u)) = 0$, it follows from [12, Lemma 5.3] that u is a polynomial. Therefore, $\lambda = 2$ and by symmetry $u(x) = q(x', x_n) + \alpha\, x_{n+1}^2$, with $q : \mathbb{R}^n \to \mathbb{R}$ a 2-homogeneous polynomial and $\alpha \in \mathbb{R}$. Furthermore, by taking into account that $\Lambda(u) = \{x_n = x_{n+1} = 0\}$ we infer that $q(x', x_n) = q_1(x')x_n + \beta\, x_n^2$, with $q_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ a 1-homogeneous polynomial and $\beta \in \mathbb{R}$. Thus, $u(x) = q_1(x')x_n + \beta\, x_n^2 + \alpha\, x_{n+1}^2$, and imposing that u solves the equation we conclude that

$$\mathrm{div}(|x_{n+1}|^a \nabla u) = \mathrm{div}\big(|x_{n+1}|^a \nabla(\beta\, x_n^2 + \alpha\, x_{n+1}^2)\big) = 0.$$

In particular, from the classification in Proposition A.1, we must have $\beta\, x_n^2 + \alpha\, x_{n+1}^2 = c\,\Phi_2$, thus implying (A.26).

2. In case (ii), we consider the tangential derivatives up to the third order $\partial_i u$, $\partial_{ij} u$ and $\partial_{ijk} u$ in directions $i, j, k \in \{1, \ldots, n-1\}$. By the regularity estimate in [17] (cf. also [24, Lemma A.2]) we deduce that $\partial_i u$, $\partial_{ij} u$ and $\partial_{ijk} u \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, |x_{n+1}|^a\, \mathcal{L}^{n+1}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$. In particular, since $\partial_{ijk} u$ is $\lambda - 3 < 0$ homogeneous, it follows that $\partial_{ijk} u \equiv 0$ for all $i, j, k \in \{1, \ldots, n-1\}$. We then infer that

$$\partial_{ij} u(x'', x_n, x_{n+1}) = \partial_{ij} u(0, x_n, x_{n+1}).$$

Being $\partial_{ij}u(0,x_n,x_{n+1})$ solution to (A.25), the analysis in Proposition A.1 implies that its homogeneity $\lambda-2$ is at least s, a condition excluded by the restriction $\lambda<2+s$. We then conclude that $\partial_{ij}u\equiv 0$ for all $i,j,\in\{1,\dots,n-1\}$, thus we get

$$\partial_i u(x'',x_n,x_{n+1})=\partial_i u(0,x_n,x_{n+1}),$$

and

$$u(x'',x_n,x_{n+1})=u(0,x_n,x_{n+1})+\sum_{i=1}^{n-1}\partial_i u(0,x_n,x_{n+1})\,x_i. \tag{A.29}$$

In particular, we infer from Proposition A.1 that the only allowed homogeneity is $\lambda=1+s$, $u(0,x_n,x_{n+1})=c_0\Psi_1(x_n,x_{n+1})$ and $\partial_i u(x'',x_n,x_{n+1})=c_i\Psi_0(x_n,x_{n+1})$, for some constants $c_i\in\mathbb{R}$ (note that all these functions solve (A.25) with contact set $\Lambda=\{x_n\le 0,x_{n+1}=0\}$). Using the explicit formulas in Proposition A.1, we conclude (A.27).

3. For case (iii), we can argue analogously as above. In particular, from the $(\lambda-3)$-homogeneity of $\partial_{ijk}u$ and $\lambda-3<0$, it follows that $\partial_{ijk}u\equiv 0$ for all $i,j,k\in\{1,\dots,n\}$. Therefore, $\partial_{ij}u$ are functions which are $(\lambda-2)$-homogeneous and depend only on x_{n+1}. By a direct computation we get from (A.25) that $\partial_{ij}u=c\,x_{n+1}^{2s}$, *i.e.* $\lambda=2+2s$: since $\lambda<2+s$, we infer that $c=0$ and $\partial_{ij}u=0$ for all $i,j=\{1,\dots,n\}$, in turn implying

$$u(x',x_{n+1})=u(0,x_{n+1})+\sum_{i=1}^{n}\partial_i u(0,x_{n+1})\,x_i. \tag{A.30}$$

By taking into account the homogeneity of u and $\partial_i u$ and (A.25) (which implies, in particular, that $u(0,x_{n+1})=0$), one then obtains (A.28). □

We are now ready to prove the general case of Proposition 8.2. Actually, we show a slightly more general result.

Proposition A.3. *Let $u:\mathbb{R}^{n+1}\to\mathbb{R}$ be a non-trivial λ-homogeneous function even w.r.to x_{n+1}. Assume that u is a weak solution of (A.25).*

(i) *If $\lambda=m\in\mathbb{N}\setminus\{0,1\}$ and $\{x\cdot e=x_{n+1}=0\}\subseteq\Lambda(u)$ for some unit vector $e\in\mathbb{R}^n\times\{0\}$, then*

$$u(x)=\sum_{k=0}^{m}p_k(x'')\,\Phi_{m-k}(x\cdot e,x_{n+1}), \tag{A.31}$$

with $p_k:\mathbb{R}^{n-1}\to\mathbb{R}$ harmonic k-homogeneous polynomial.

(ii) *If $\lambda=m+s$, with $m\in\mathbb{N}\setminus\{0\}$, and $\{x\cdot e=x_{n+1}=0\}\subseteq\Lambda(u)$ for some unit vector $e\in\mathbb{R}^n\times\{0\}$, then*

$$u(x)=\sum_{k=0}^{m}p_k^+(x'')\,\Psi_{m-k}(x\cdot e,x_{n+1})+\sum_{k=0}^{m}p_k^-(x'')\,\Psi_{m-k}(-x\cdot e,x_{n+1}), \tag{A.32}$$

with $p_k:\mathbb{R}^{n-1}\to\mathbb{R}$ harmonic k-homogeneous polynomial.

(iii) *If $\lambda=m+2s$, with $m\in\mathbb{N}$, and $\Lambda(u)=\{x_{n+1}=0\}$, then*

$$u(x)=|x_{n+1}|^{2s}\sum_{k=0}^{\lfloor m/2\rfloor}\gamma_k\,x_{n+1}^{2k}\Delta^k p(x') \tag{A.33}$$

with $p:\mathbb{R}^n\to\mathbb{R}$ any m-homogeneous polynomial and $\gamma_k:=\frac{(-1)^k}{4^k k!\,(1+s)_k}$.

Moreover, if u is a solution to the thin obstacle problem (1.1), then in case (i), respectively (ii), u turns out to be a positive multiple of $h_{2m}(x\cdot e,x_{n+1})$, respectively $h_{2m-1+s}(\pm x\cdot e,x_{n+1})$.

Proof. Without loss of generality we assume that $e=e_n$. The proof proceeds by induction on $m\in\mathbb{N}$, with starting step provided by Proposition A.2.

The cases (i) and (ii) can be treated by the same argument. We consider the horizontal partial derivatives $\partial_{x_j}u$ for $j\in\{1,\dots,n-1\}$. By the regularity estimate in [17] we have that $\partial_{x_j}u\in$

$H^1_{\rm loc}(\mathbb{R}^{n+1}, |x_{n+1}|^a\,\mathcal{L}^{n+1}) \cap L^\infty_{\rm loc}(\mathbb{R}^{n+1})$ are solutions to (8.3) with $\{x_n = x_{n+1}\} \subseteq \Lambda(\partial_i u)$ and homogeneity $\lambda - 1 = m-1$ or $\lambda = m-1+s$, according to the two cases. Using the inductive hypothesis $\partial_i u = \sum_{k=0}^{m-1} p_{i,k}\Phi_{m-1-k}$ for $\lambda = m$ or $\partial_i u = \sum_{k=0}^{m-1} p^+_{i,k}\Psi_{m-1-k}(\cdot,\cdot) + \sum_{k=0}^{m-1} p^-_{i,k}\Psi_{m-1-k}(-\cdot,\cdot)$ for $\lambda = m+s$, for some harmonic k-homogeneous polynomials $p_{i,k}$ and $p^\pm_{i,k}$. Therefore, we infer that

$$\begin{aligned} u(x) &= u(0,x_n,x_{n+1}) + \int_0^1 \sum_{i=1}^{n-1} \partial_{x_i} u(tx'', x_n, x_{n+1})\, x_i \,\mathrm{d}t \\ &= u(0,x_n,x_{n+1}) + \int_0^1 \sum_{i=1}^{n-1}\sum_{k=0}^{m-1} t^k p_{i,k}(x'')\Phi_{m-1-k}(x_n,x_{n+1})\, x_i\,\mathrm{d}t \\ &= u(0,x_n,x_{n+1}) + \sum_{k=0}^{m-1} p_k(x'')\Phi_{m-1-k}(x_n,x_{n+1}), \end{aligned}$$

and similarly

$$u(x) = u(0,x_n,x_{n+1}) + \sum_{k=0}^{m-1} p^+_k(x'')\Psi_{m-1-k}(x_n,x_{n+1}) + \sum_{k=0}^{m-1} p^-_k(x'')\Psi_{m-1-k}(-x_n,x_{n+1}),$$

with $k\, p^{(\pm)}_k(x'') := \sum_{i=1}^{n-1} p^{(\pm)}_{i,k}(x'')x_i$. Using the equation (A.25) (in particular, recall that Φ_l, Ψ_l are solutions of (A.25)), we deduce that the polynomials $p_k, p_k^\pm$ are harmonic and $u(0,x_n,x_{n+1})$ is itself a solution (*i.e.* $u(0,x_n,x_{n+1}) = c\,\Phi_m(x_n,x_{n+1})$ or $u(0,x_n,x_{n+1}) = c_1\,\Psi_m(x_n,x_{n+1}) + c_2\,\Psi_m(-x_n,x_{n+1})$ for some $c, c_1, c_2 \in \mathbb{R}$), thus concluding the proof for the cases (i) and (ii).

In case (iii) with $\lambda = m+2s$, we consider instead all the horizontal derivatives of u and use the inductive hypothesis (A.33) in the form

$$\partial_i u(x) = |x_{n+1}|^{2s} \sum_{k=0}^{\lfloor\frac{(m-1)}{2}\rfloor} x^{2k}_{n+1} q_{i,m-1-2k}(x') \quad \forall\, i \in \{1,\dots,n\},$$

where q^i_{m-2k} are $(m-1-2k)$-homogeneous polynomials. Therefore,

$$\begin{aligned} u(x) &= u(0,x_{n+1}) + |x_{n+1}|^{2s}\int_0^1 \sum_{i=1}^{n}\sum_{k=0}^{\lfloor\frac{(m-1)}{2}\rfloor} x^{2k}_{n+1} q_{i,m-1-2k}(tx')\, x_i\,\mathrm{d}t \\ &= u(0,x_{n+1}) + |x_{n+1}|^{2s}\sum_{k=0}^{\lfloor\frac{(m-1)}{2}\rfloor} x^{2k}_{n+1} q_{m-2k}(x), \end{aligned}$$

with $(m-2k)q_{m-2k}(x) = \sum_{i=1}^n q_{i,m-1-2k}(x')x_i$. Taking into account the homogeneity of u, we infer that $u(0,x_{n+1}) = c|x_{n+1}|^{m+2s}$ and the exact form for the polynomials q_k in (A.33) follows by using the equation (A.25).

Finally, we discuss the case of solution to the obstacle problem (1.1). In case (i), the unilateral condition $u \geq 0$ on B_1' implies that

$$u(x'', x_n, 0) = \sum_{k=1}^{m} p_k(x'')\, x_n^{m-k}\,\Phi_{m-k}(1,0) + p_0\,\Phi_m(x_n,0) \geq 0 \quad \forall\, x'' \in \mathbb{R}^{n-1}\times\{0\}.$$

This implies that the polynomials p_k with $k \in \{1,\dots,m\}$ are all zero. Let, indeed, $j := \min\{k \in \{1,\dots,m\} : p_k \not\equiv 0\}$ and divide u by $x_n^j > 0$: by taking the limit as $x_n \downarrow 0$ we infer that p_j is a constant sign homogeneous harmonic polynomial, which holds only if $p_j \equiv 0$, thus giving a contradiction. Therefore, we conclude $u(x) = p_0\Phi_m(x_n,x_{n+1})$ with $p_0 > 0$ for solutions to the obstacle problem.

For the case (ii), by the same argument we deduce that all polynomials $p^\pm_k \equiv 0$ for $k \in \{1,\dots,m\}$, and therefore $u(x) = p_0^+\Psi_m(x_n,x_{n+1}) + p_0^-\Psi_m(-x_n,x_{n+1})$ with $p_0^\pm \geq 0$. Since u is a function of two variables, the conclusion follows now from Lemma 5.3. □

References

[1] Almgren, Frederick J., Jr. *Almgren's big regularity paper.* World Scientific Monograph Series in Mathematics , 1. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[2] L. Ambrosio, N. Fusco, D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp.

[3] J. Andersson. Optimal regularity for the Signorini problem and its free boundary. *Invent. Math.* 204 (2016), no. 1, 1–82.

[4] I. Athanasopoulos, L. A. Caffarelli. Optimal regularity of lower dimensional obstacle problems. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 310 (2004), 49–66, 226; translation in *J. Math. Sci. (N. Y.)*, 132 (2006), no. 3, 274–284.

[5] I. Athanasopoulos, L. A. Caffarelli, S. Salsa. The structure of the free boundary for lower dimensional obstacle problems. *Amer. J. Math.* 130 (2008), no. 2, 485–498.

[6] J. Azzam, X. Tolsa. Characterization of n-rectifiability in terms of Jones' square function: Part II. *Geom. Funct. Anal.* 25 (2015), no. 5, 1371–1412.

[7] B. Barrios, A. Figalli, X. Ros-Oton. Global regularity for the free boundary in the obstacle problem for the fractional Laplacian. ArXiv:1506.04684

[8] L. A. Caffarelli. The regularity of free boundaries in higher dimensions. *Acta Math.* 139 (1977), no. 3-4, 155–184.

[9] L. A. Caffarelli. Further regularity for the Signorini problem. *Comm. Partial Differential Equations* 4 (1979), 1067–1075.

[10] L. A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.* 4 (1998), no. 4-5, 383–402.

[11] L. A. Caffarelli, L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* 32 (2007), no. 7-9, 1245–1260.

[12] L. A. Caffarelli, S. Salsa, L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.* 171 (2008), no. 2, 425–461.

[13] G. David, T. Toro. Reifenberg parameterizations for sets with holes. *Mem. Amer. Math. Soc.* 215 (2012), no. 1012, vi+102 pp.

[14] D. De Silva, O. Savin. C^∞ regularity of certain thin free boundaries. *Indiana Univ. Math. J.* 64 (2015), no. 5, 1575–1608.

[15] C. De Lellis, E. Spadaro. Regularity of area minimizing currents III: blow-up. *Ann. of Math.* (2) 183 (2016), no. 2, 577–617.

[16] C. De Lellis, A. Marchese, E. Spadaro, D. Valtorta. Rectifiability and upper Minkowski bounds for singularities of harmonic Q-valued maps. ArXiv:1612.01813.

[17] E. Fabes, C. Kenig, R. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* 7 (1982), no. 1, 77–116.

[18] M. Focardi, M. S. Gelli, E. Spadaro. Monotonicity formulas for obstacle problems with Lipschitz coefficients. *Calc. Var. Partial Differential Equations* 54 (2015), no. 2, 1547–1573.

[19] M. Focardi, A. Marchese, E. Spadaro. Improved estimate of the singular set of Dir-minimizing Q-valued functions via an abstract regularity result. *J. Funct. Anal.* 268 (2015), no. 11, 3290–3325.

[20] M. Focardi, E. Spadaro. An epiperimetric inequality for the fractional obstacle problem. *Adv. Differential Equations* 21 (2016), no. 1-2, 153–200.

[21] J. Frehse. Two-dimensional variational problems with thin obstacles. *Math. Z.* 143 (1975), 279–288.

[22] J. Frehse. On Signorini's problem and variational problems with thin obstacles. *Ann. Scuola Norm. Sup. Pisa* 4 (1977), 343–362.

[23] N. Garofalo, A. Petrosyan. Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Invent. Math.*, 177 (2009), no. 2, 415–461.

[24] N. Garofalo, A. Petrosyan, C. Pop, M. Smit Vega Garcia Regularity of the free boundary for the obstacle problem for the fractional Laplacian with drift. ArXiv:1509.06228.

[25] F. Geraci. The classical obstacle problem for nonlinear variational energies and related problems. PhD thesis, University of Firenze (2016).

[26] E. Hopf. Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung. (German) Math. Z. 34 (1932), no. 1, 194–233.

[27] P. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.* 102 (1990), no. 1, 1–15.

[28] D. Kinderlehrer. The smoothness of the solution of the boundary obstacle problem. *J. Math. Pures Appl.* 60 (1981), 193–212.

[29] H. Koch, A. Petrosyan, W. Shi. Higher regularity of the free boundary in the elliptic Signorini problem. *Nonlinear Anal.* 126 (2015), 3–44.

[30] R. Monneau. On the number of singularities for the obstacle problem in two dimensions. *J. Geom. Anal.* 13 (2003), no. 2, 359–389.

[31] A. Naber, D. Valtorta. Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps. *Ann. of Math. (2)* 185 (2017), no. 1, 131–227.

[32] A. Naber, D. Valtorta. The Singular Structure and Regularity of Stationary and Minimizing Varifolds. *preprint* arXiv:1505.03428

[33] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.14 of 2016-12-21. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds.

[34] X. Ros-Oton, J. Serra Boundary regularity for fully nonlinear integro-differential equations. *Duke Math. J.* 165 (2016), no. 11, 2079–2154.

[35] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* 60 (2007), no. 1, 67–112.

[36] N. N. Uraltseva Hölder continuity of gradients of solutions of parabolic equations with boundary conditions of Signorini type. *Dokl. Akad. Nauk SSSR* 280 (1985), 563–565.

[37] N. N. Uraltseva On the regularity of solutions of variational inequalities. (Russian) *Uspekhi Mat. Nauk* 42 (1987), no. 6(258), 151–174, 248.

[38] G. S. Weiss. A homogeneity improvement approach to the obstacle problem. *Invent. Math.*, 138 (1999), no. 1, 23–50.

DiMaI, Università degli Studi di Firenze
Current address: Viale Morgagni 67/A, 50134 Firenze (Italy)
E-mail address: focardi@math.unifi.it

Universität Leipzig
Current address: Augustusplatz 10, 04109 Leipzig (Germany)
E-mail address: Emanuele.Spadaro@math.uni-leipzig.de