A DIRECT APPROACH TO PLATEAU’S PROBLEM IN ANY CODIMENSION

G. DE PHILIPPIS, A. DE ROSA, AND F. GHIRALDIN

Abstract. This paper proposes a direct approach to solve the Plateau’s problem in codimension higher than one. The problem is formulated as the minimization of the Hausdorff measure among a family of \(d\)-rectifiable closed subsets of \(\mathbb{R}^n\): following the previous work [DGM14], the existence result is obtained by a compactness principle valid under fairly general assumptions on the class of competitors. Such class is then specified to give meaning to boundary conditions. We also show that the obtained minimizers are regular up to a set of dimension less than \((d-1)\).

1. Introduction

Plateau’s problem consists in looking for a surface of minimal area among those surfaces spanning a given boundary. A considerable amount of effort in Geometric Measure Theory during the last fifty years has been devoted to provide generalized concepts of surface, area and of “spanning a given boundary”, in order to apply the direct methods of the calculus of variations to the Plateau’s problem. In particular we recall the notions of sets of finite perimeter [De54, De55], of currents [FF60] and of varifolds [All72, All75, Alm68], introduced respectively by De Giorgi, Federer, Fleming, Almgren and Allard. A more “geometric” approach was proposed by Reifenberg in [Rei60], where Plateau’s problem was set as the minimization of Hausdorff \(d\)-dimensional measure among compact sets and the notion of spanning a given boundary was given in term of inclusions of homology groups.

Any of these approach has some drawbacks: in particular, not all the “reasonable” boundaries can be obtained by the above notions and not always the solutions are allowed to have the type of singularities observed in soap bubbles (the so called Plateau’s laws). Recently in [HP13] Harrison and Pugh, see also [Har14], proposed a new notion of spanning a boundary, which seems to include reasonable physical boundaries and they have been able to show, in the codimension one case, existence of least area surfaces spanning a given boundary.

In the recent paper [DGM14], De Lellis, Maggi and the third author have proposed a direct approach to the Plateau’s problem, based on the “elementary” theory of Radon measures and on a deep result of Preiss concerning rectifiable measures. Roughly speaking they showed, in the codimension one case, that every time one has a class which contains “enough” competitors (namely the cone and the cup competitors, see [DGM14, Definition 1]) it is always possible to show that the infimum of the Plateau’s problem is achieved by the area of a rectifiable set. They then applied this result to provide a new proof of Harrison and Pugh theorem as well as to show the existence of sliding minimizers, a new notion of minimal sets proposed by David in [Dav14, Dav13] and inspired by Almgren’s \((M, 0, \infty)\), [Alm76].

In this note, we extend the result [DGM14] to any codimension. More precisely, we prove that every time the class of competitors for the Plateau’s Problem consists of rectifiable sets and it is closed by Lipschitz deformations, it is always possible to show that the infimum is achieved by a compact set \(K\) which is, away from the “boundary”, an analytic manifold outside a closed set of Hausdorff dimension at most \((d-1)\), see Theorem [1.3] below for the precise statement. We then apply this result to provide existence of sets spanning a given boundary according to
the natural generalization of the notion introduced by Harrison and Pugh, Theorem 1.3, and to show the existence of sliding minimizers in any codimension, Theorem 1.8.

Although the general strategy of the proof is the same of [DGM14], some non-trivial modifications have to be done in order to deal with sets of any co-dimension. In particular, with respect to [DGM14], we use a different notion of “good class”, the main reason being the following: one of the key step of the proof of our main result consists in showing a precise density lower bound for the measure obtained as limit of the sequence of Radon measures naturally associated to a minimizing sequence \((K_j)\), see Steps 1 and 4 in the proof of Theorem 1.3. In order to obtain such a lower bound, instead of relying on relative isoperimetric inequalities on the sphere as in [DGM14] (which are peculiar of the co-dimension one case) we use the deformation theorem of David and Semmes in [DS00] to obtain suitable competitors, following a strategy already introduced by Federer and Fleming for rectifiable currents, see [FF60] and [Alm76]. Moreover since our class is essentially closed by Lipschitz deformations, we are actually able to prove that any set achieving the infimum is a stationary varifold and that, in addition, it is smooth outside a closed set of relative co-dimension one (this does not directly follows by Allard’s regularity theorem, see Step 7 in the proof of Theorem 1.3). Simple examples show that this regularity is actually optimal.

In order to precisely state our main results, let us introduce some notations and definitions, referring to Section 2 for more details. We will always work in \(\mathbb{R}^n\) and \(1 \leq d \leq n\) will always be an integer number, we recall that a set \(K\) is said to be \(d\)-rectifiable if it can be covered up to an \(H^d\) negligible set by countably many \(C^1\) manifolds, see [Sim83, Chapter 3], where \(H^d\) is the \(d\)-dimensional Hausdorff measure. We also let \(\text{Lip}(\mathbb{R}^n)\) be the space of Lipschitz maps in \(\mathbb{R}^n\).

**Definition 1.1** (Lipschitz deformations). Given a ball \(B_{x,r}\), we let \(\mathcal{D}(x,r)\) be the set of functions \(\varphi : \mathbb{R}^n \to \mathbb{R}^n\) such that \(\varphi(z) = z\) in \(\mathbb{R}^n \setminus B_{x,r}\) and which are smoothly isotopic to the identity inside \(B_{x,r}\), namely those for which there exists a smooth isotopy \(\lambda : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n\) such that

\[
\lambda(0, \cdot) = \text{Id}, \quad \lambda(1, \cdot) = \varphi, \quad \lambda(t, h) = h \quad \forall (t, h) \in [0,1] \times (\mathbb{R}^n \setminus B_{x,r}) \quad \text{and} \quad \lambda(t, \cdot) \text{ is a diffeomorphism of } \mathbb{R}^n \forall t \in [0,1].
\]

We finally set \(\mathcal{D}(x,r) := \overline{\mathcal{D}(x,r)}^{C^0} \cap \text{Lip}(\mathbb{R}^n)\), the intersection of the Lipschitz maps with the closure of \(\mathcal{D}(x,r)\) with respect to the uniform topology.

The following definition describes the properties required on comparison sets: the key property for \(K'\) to be a competitor of \(K\) is that \(K'\) is close in energy to sets obtained from \(K\) via deformation maps as in Definition 1.1. This allows a larger flexibility on the choice of the admissible sets, since a priori \(K'\) might not belong to the competition class.

**Definition 1.2** (Deformed competitors and good class). Let \(H \subset \mathbb{R}^n\) be closed, \(K \subset \mathbb{R}^n \setminus H\) relatively closed countably \(\mathcal{H}^d\)-rectifiable and \(B_{x,r} \subset \mathbb{R}^n \setminus H\). A deformed competitor for \(K\) in \(B_{x,r}\) is any set of the form

\[
\varphi(K) \quad \text{where} \quad \varphi \in \mathcal{D}(x,r).
\]

Given a family \(\mathcal{P}(H)\) of relatively closed \(d\)-rectifiable subsets \(K \subset \mathbb{R}^n \setminus H\), we say that \(\mathcal{P}(H)\) is a good class if for every \(K \in \mathcal{P}(H)\), for every \(x \in K\) and for a.e. \(r \in (0, \text{dist}(x,H))\)

\[
\inf \{ \mathcal{H}^d(J) : J \in \mathcal{P}(H), J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}} \} \leq \mathcal{H}^d(L) \tag{1.1}
\]

whenever \(L\) is any deformed competitor for \(K\) in \(B_{x,r}\).

Once we fix a closed set \(H\), we can formulate Plateau’s problem in the class \(\mathcal{P}(H)\):

\[
m_0 := \inf \{ \mathcal{H}^d(K) : K \in \mathcal{P}(H) \}. \tag{1.2}
\]

We will say that a sequence \((K_j) \subset \mathcal{P}(H)\) is a minimizing sequence if \(\mathcal{H}^d(K_j) \downarrow m_0\). The following theorem is our main result and establishes the behavior of minimizing sequences.
Theorem 1.3. Let $H \subset \mathbb{R}^n$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau’s problem (1.2) is finite and let $(K_j) \subset \mathcal{P}(H)$ be a minimizing sequence. Then, up to subsequences, the measures $\mu_j := \mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to a measure $\mu = \mathcal{H}^d \llcorner K$, where $K = \text{spt} \mu \cap H$ is a countably $\mathcal{H}^d$-rectifiable set. Furthermore:

(a) the integral varifold naturally associated to $\mu$ is stationary in $\mathbb{R}^n \setminus H$;
(b) $K$ is a real analytic submanifold outside a relatively closed set $\Sigma \subset H$ with $\text{dim}_H(\Sigma) \leq d - 1$.

In particular, $\liminf \mathcal{H}^d(K_j) \geq \mathcal{H}^d(K)$ and if $K \in \mathcal{P}(H)$, then $K$ is a minimum for (1.2).

We wish to apply Theorem 1.3 to two definitions of boundary conditions. The first one is the natural generalization of the one considered in [HP13].

Definition 1.4. Let $H$ be a closed set in $\mathbb{R}^n$.

Let us consider the family

$$C_H = \{ \gamma : S^{n-d} \to \mathbb{R}^n \setminus H : \gamma \text{ is a smooth embedding of } S^{n-d} \text{ into } \mathbb{R}^n \}.$$

We say that $C \subset C_H$ is closed by isotopy (with respect to $H$) if $C$ contains all elements $\gamma' \in C_H$ belonging to the same smooth isotopy class $[\gamma]$ in $\mathbb{R}^n \setminus H$ of any $\gamma \in C$, see [Hir91] Ch. 8. Given $C \subset C_H$ closed by isotopy, we say that a relatively closed subset $K$ of $\mathbb{R}^n \setminus H$ is a $C$-spanning set of $H$ if

$$K \cap \gamma \neq \emptyset \text{ for every } \gamma \in C.$$

We denote by $F(H, C)$ the family of countably $\mathcal{H}^d$-rectifiable sets which are $C$-spanning sets of $H$.

We can prove the following closure property for the class $F(H, C)$:

Theorem 1.5. Let $H$ be closed in $\mathbb{R}^n$ and $C$ be closed by isotopy with respect to $H$, then:

(a) $F(H, C)$ is a good class in the sense of Definition 1.2.
(b) If the infimum (1.2) corresponding to $\mathcal{P}(H) = F(H, C)$ is finite, then the set $K$ provided by Theorem 1.3 belongs to $F(H, C)$. In particular the Plateau’s problem in the class $F(H, C)$ has a solution.

The second type of boundary condition we want to consider is the one related to the notion of “sliding minimizers” introduced by David in [Dav14, Dav13].

Definition 1.6 (Sliding minimizers). Let $H \subset \mathbb{R}^n$ be closed and $K_0 \subset \mathbb{R}^n \setminus H$ be relatively closed. We denote by $\Sigma(H)$ the family of Lipschitz maps $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a continuous map $\Phi : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(1, \cdot) = \varphi$, $\Phi(0, \cdot) = \text{Id}$ and $\Phi(t, H) \subset H$ for every $t \in [0, 1]$. We then define

$$\mathcal{A}(H, K_0) = \{ K : K = \varphi(K_0) \text{ for some } \varphi \in \Sigma(H) \}$$

and say that $K_0$ is a sliding minimizer if $\mathcal{H}^d(K_0) = \inf \{ \mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0) \}$.

Remark 1.7. For every $K_0 \subset \mathbb{R}^n \setminus H$ relatively closed and $d$-rectifiable, $\mathcal{A}(H, K_0)$ is a good class in the sense of Definition 1.2 since $\text{D}(x, r) \subset \Sigma(H)$ for every $B_{x, r} \subset \mathbb{R}^n \setminus H$.

Applying Theorem 1.3 to the contest of sliding minimizers we obtain the following result which is the analogous of [DGM14, Theorem 7] in any codimension. Here and in the following $U_{\delta}(E)$ denotes the $\delta$-neighborhood of a set $E \subset \mathbb{R}^n$.

Theorem 1.8. Assume that

(i) $K_0$ is bounded $d$-rectifiable with $\mathcal{H}^d(K_0) < \infty$;
(ii) $\mathcal{H}^d(H) = 0$ and for every $\eta > 0$ there exist $\delta > 0$ and $\Pi \in \Sigma(H)$ such that $\text{Lip} \Pi \leq 1 + \eta$, $\Pi(U_{\delta}(H)) \subset H$. 

Then, given any minimizing sequence \((K_j)\) in the Plateau’s problem corresponding to \(\mathcal{P}(H) = \mathcal{A}(H, K_0)\) and any set \(K\) as in Theorem 1.3, we have
\[
\inf \{ \mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0) \} = \mathcal{H}^d(K) = \inf \{ \mathcal{H}^d(J) : J \in \mathcal{A}(H, K) \}.
\]
In particular \(K\) is a sliding minimizer.

The paper is structured as follows, in Section 2 we will recall some basic definitions and recall some known theorems we are going to use, in particular Preiss rectifiability criterion and a version of the deformation theorem due to David and Semmes. In Section 3 we prove Theorem 1.3 and in Section 4 we prove Theorems 1.5 and 1.8.

**Acknowledgements.** The authors are grateful to Camillo De Lellis, Francesco Maggi and Emanuele Spadaro for many interesting comments and suggestions. This work has been supported by ERC 306247 Regularity of area-minimizing currents and by SNF 146349 Calculus of variations and fluid dynamics.

2. Notation and preliminaries

We are going to use the following notations: \(Q_{x,l}\) denotes the closed cube centered in \(x\), with edge length \(l\); moreover we set
\[
R_{x,a,b} := x + \left[ -\frac{a}{2}, \frac{a}{2} \right]^d \times \left[ -\frac{b}{2}, \frac{b}{2} \right]^{n-d} \quad \text{and} \quad B_{x,r} := \{ y \in \mathbb{R}^n : |y-x| < r \}.
\]
When cubes, rectangles and balls are centered in the origin, we will simply write \(Q_l\), \(R_{a,b}\) and \(B_r\). Cubes and balls in the subspace \(\mathbb{R}^d \times \{0\}^{n-d}\) are denoted with \(Q_{x,l}^d\) and \(B_{x,r}^d\) respectively. We also let \(\omega_d\) be the Lebesgue measure of the unit ball in \(\mathbb{R}^d\).

Let us recall the following deep structure result for Radon measures due to Preiss [Pre87, Del08] which will play a key role in the proof of Theorem 1.3.

**Theorem 2.1.** Let \(d\) be an integer and \(\mu\) a locally finite measure on \(\mathbb{R}^n\) such that the \(d\)-density
\[
\theta(x) := \lim_{r \to 0} \frac{\mu(B_{x,r})}{\omega_d r^d}
\]
exists and satisfies \(0 < \theta(x) < +\infty\) for \(\mu\)-a.e. \(x\). Then \(\mu = \theta \mathcal{H}^d \mathbb{1}_K\), where \(K\) is a countably \(\mathcal{H}^d\)-rectifiable set.

In order to apply Preiss’ Theorem we will rely on the monotonicity formula for minimal surfaces, which roughly speaking can be obtained by comparing the given minimizer with a cone. To this aim let us introduce the following definition:

**Definition 2.2** (Cone competitors). In the setting of Definition 1.2 the cone competitor for \(K\) in \(B_{x,r}\) is the following set
\[
C_{x,r}(K) = (K \setminus B_{x,r}) \cup \{ \lambda x + (1 - \lambda) z : z \in K \cap \partial B_{x,r}, \lambda \in [0, 1] \}.
\]  

Let us note that in general a cone competitor in \(B_{x,r}\) is not a deformed competitor in \(B_{x,r}\). On the other hand as in [DGM14] we can show that:

**Lemma 2.3.** Given a good class \(\mathcal{P}(H)\) in the sense of Definition 1.2, for any \(K \in \mathcal{P}(H)\) countably \(\mathcal{H}^d\)-rectifiable and for every \(x \in K\), the set \(K\) verifies for a.e. \(r \in (0, \text{dist}(x, H))\):
\[
\inf \{ \mathcal{H}^d(J) : J \in \mathcal{P}(H), J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}} \} \leq \mathcal{H}^d(C_{x,r}(K))
\]

**Proof.** Without loss of generality let us consider balls \(B_r\) centered at 0 with \(B_r \subset \subset \mathbb{R}^n \setminus H\). We assume in addition that \(K \cap \partial B_r\) is \(\mathcal{H}^{d-1}\)-rectifiable with \(\mathcal{H}^{d-1}(K \cap \partial B_r) < \infty\) and that \(r\) is a
Lebesgue point of \( t \in (0, \infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t) \). All these conditions are fulfilled for a.e. \( r \) and, again by scaling, we can assume that \( r = 1 \) and use \( B \) instead of \( B_1 \). For \( s \in (0, 1) \) let us set
\[
\varphi_s(r) = \begin{cases} 
0, & r \in [0, 1-s), \\
\frac{r-(1-s)}{s}, & r \in [1-s, 1], \\
r, & r \geq 1,
\end{cases}
\]
and \( \phi_s(x) = \varphi_s(|x|) \frac{x}{|x|} \) for \( x \in \mathbb{R}^n \). In this way, one easily checks that \( \phi_s : \mathbb{R}^n \to \mathbb{R}^n \in D(0,1) \).

Since \( \phi_s(K \cap B_1) = \{0\}, \) we need to show that
\[
\limsup_{s \to 0^+} \mathcal{H}^d(\phi_s(K \cap (B \setminus B_{1-s}))) \leq \frac{\mathcal{H}^{d-1}(K \cap \partial B)}{d} = \mathcal{H}^d(C_{x,r}(K)) .
\]

Let \( x_0 \in K \cap \partial B_t \) and let us fix an orthonormal base \( \nu_1, \ldots, \nu_d \) of the approximate tangent space \( T_{x_0}K \) such that \( \nu_i \in T_{x_0}K \cap T_{x_0} \partial B_t \) for \( i \leq d-1 \). Let
\[
J^K_d \phi_s = \left( \bigwedge D\phi_s(T_{x_0}K) \right) = |D\phi_s(\nu_1) \wedge \cdots \wedge D\phi_s(\nu_d)|
\]
be the \( d \)-dimensional tangential Jacobian of \( \phi_s \) with respect to \( K \). A simple computation shows that
\[
J^K_d \phi_s(x) \leq \left( \frac{\varphi_s(|x|)}{|x|} \right)^d + |\nu_d \cdot \hat{x}| \varphi'_s(|x|) \left( \frac{\varphi_s(|x|)}{|x|} \right)^{d-1}
\leq 1 + |\nu_d \cdot \hat{x}| \varphi'_s(|x|) \left( \frac{\varphi_s(|x|)}{|x|} \right)^{d-1}, \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in K .
\]

Here \( \hat{x} = x/|x| \) and in the last inequality we have exploited that \( \varphi(r) \leq r \) for \( r \in [1-s, 1] \). Using that \( |\nu_d \cdot \hat{x}| \) is the tangential co-area factor of the map \( f(x) = |x| \), we find with the aid of the area and co-area formulas,
\[
\mathcal{H}^d(\phi_s(K \cap (B \setminus B_{1-s}))) = \int_{K \cap (B \setminus B_{1-s})} J^K_d \phi_s d\mathcal{H}^d
= \int_{K \cap (B \setminus B_{1-s}) \setminus \{ |\nu_d \cdot \hat{x}| \neq 0 \}} J^K_d \phi_s d\mathcal{H}^d + \int_{K \cap (B \setminus B_{1-s}) \setminus \{ |\nu_d \cdot \hat{x}| = 0 \}} J^K_d \phi_s d\mathcal{H}^d
\leq \int_{1-s}^1 dt \int_{K \cap \partial B_t \setminus \{ |\nu_d \cdot \hat{x}| = 0 \}} J^K_d \phi_s \frac{1}{|\nu_d \cdot \hat{x}|} d\mathcal{H}^{d-1} + \mathcal{H}^d(K \cap (B \setminus B_{1-s}) \setminus \{ |\nu_d \cdot \hat{x}| = 0 \}) ,
\]

since \( |J^K_d \phi_s| \leq 1 \) where \( |\nu_d \cdot \hat{x}| \leq 1 \). Since
\[
\lim_{s \to 0} \mathcal{H}^d(K \cap (B \setminus B_{1-s})) = 0 .
\]
the second term in (2.3) can be ignored. Moreover, being \( t = 1 \) a Lebesgue point of \( t \in (0, \infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t) \), we have
\[
\lim_{s \to 0^+} \frac{1}{s} \int_{1-s}^1 \mathcal{H}^{d-1}(K \cap \partial B_t) - \mathcal{H}^{d-1}(K \cap \partial B)\, dt = 0 .
\]

Thanks to this and to the estimate (2.2) we infer from (2.3) that
\[
\limsup_{s \to 0^+} \mathcal{H}^d(\varphi_s(K \cap B)) \leq \mathcal{H}^{d-1}(K \cap \partial B) \limsup_{s \to 0^+} \frac{1}{s} \int_{1-s}^1 \left( \frac{\varphi_s(t)}{t} \right)^{d-1} dt = \frac{\mathcal{H}^{d-1}(K \cap \partial B)}{d} ,
\]
as required. \( \square \)
The second key result we are going to use is a deformation theorem for closed sets due to David and Semmes [DS00], analogous to the one for rectifiable currents [Sim83, Fed69]. We provide a slightly extended statement for the sake of forthcoming proofs.

Before stating the theorem, let us introduce some further notation. Given a closed cube $Q = Q_{x,l}$ and $\varepsilon > 0$, we cover $Q$ with closed smaller cubes with edges length $\varepsilon \ll l$, non empty intersection with $\text{Int}(Q)$ and such that the decomposition is centered in $x$ (i.e. one of the subcubes is centered in $x$). The family of this smaller cubes is denoted $\Lambda_\varepsilon(Q)$. We set

$$C_1 := \bigcup \{ T \cap Q : T \in \Lambda_\varepsilon(Q), T \cap \partial Q \neq \emptyset \},$$

$$C_2 := \bigcup \{ T \in \Lambda_\varepsilon(Q) : T \not\subset C_1, T \cap \partial C_1 \neq \emptyset \},$$

$$Q^1 := Q \setminus (C_1 \cup C_2)$$

and consequently

$$\Lambda_\varepsilon(Q^1 \cup C_2) := \{ T \in \Lambda_\varepsilon(Q) : T \subset (Q^1 \cup C_2) \}$$

For each nonnegative integer $m \leq n$, let $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ denote the collection of all $m$-dimensional faces of cubes in $\Lambda_\varepsilon(Q^1 \cup C_2)$ and $\Lambda_{\varepsilon,m}^*(Q^1 \cup C_2)$ be the set of the elements of $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ which are not contained in $\partial(Q^1 \cup C_2)$. We also let $S_{\varepsilon,m}(Q^1 \cup C_2) := \bigcup \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ be the the $m$-skeleton of order $\varepsilon$ in $Q^1 \cup C_2$.

**Theorem 2.4.** Let $r > 0$ and $E$ be a compact subset of $Q$ such that $\mathcal{H}^d(E) < +\infty$ and $Q \subset B_{x_0,r}$.

There exists a map $\Phi_{\varepsilon,E} \in \mathcal{D}(x_0, r)$ satisfying the following properties:

1. $\Phi_{\varepsilon,E}(x) = x$ for $x \in \mathbb{R}^n \setminus (Q^1 \cup C_2)$;
2. $\Phi_{\varepsilon,E}(x) = x$ for $x \in S_{\varepsilon,d-1}(Q^1 \cup C_2)$;
3. $\Phi_{\varepsilon,E}(E) \subset S_{\varepsilon,d}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2)$;
4. $\Phi_{\varepsilon,E}(T) \subset T$ for every $T \in \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$, with $m = d, \ldots, n$;
5. either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) = 0$ or $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) = \mathcal{H}^d(T)$, for every $T \in \Lambda_{\varepsilon,d}^*(Q^1)$;
6. $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) \leq k_1 \mathcal{H}^d(E \cap T)$ for every $T \in \Lambda_{\varepsilon}(Q^1 \cup C_2)$;

where $k_1$ depends only on $n$ and $d$ (but not on $\varepsilon$).

**Proof.** Proposition 3.1 in [DS00] provides a map $\Phi_{\varepsilon,E} \in \mathcal{D}(x_0, r)$ satisfying properties (1)-(4) and (6). We want to set

$$\Phi_{\varepsilon,E} := \Psi \circ \Phi_{\varepsilon,E},$$

where $\Psi$ will be defined below. We first define $\Psi$ on every $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ distinguishing two cases

(a) if either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) = 0$ or $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) = \mathcal{H}^d(T)$ or $T \not\subset \Lambda_{\varepsilon,d}^*(Q^1)$, then we set $\Psi(T) = \text{Id}$;

(b) otherwise since $\Phi_{\varepsilon,E}(E)$ is compact, there exists $y_T \in T$ and $\delta_T > 0$ such that $B_{\delta_T}(y_T) \cap \Phi_{\varepsilon,E}(E) = \emptyset$; we define

$$\Psi(T)(x) = x + \alpha(x - y_T) \min \left\{ 1, \frac{|x - y_T|}{\delta_T} \right\},$$

where $\alpha > 0$ such that the point $x + \alpha(x - y_T) \in (\partial T) \times \{0\}^{n-d}$.

The second step is to define $\Psi$ on every $T' \in \Lambda_{\varepsilon,d+1}(Q^1 \cup C_2)$. Without loss of generality we can assume $T'$ centered in 0. We divide $T'$ in pyramids $P_{T,T'}$ with base $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ and vertex 0. Assuming $T \subset \{x_{d+1} = -\frac{\varepsilon}{2}, x_{d+2}, \ldots, x_n = 0\}$ and $T' \subset \{x_{d+2}, \ldots, x_n = 0\}$, we set

$$\Psi_{P_{T,T'}}(x) = \frac{-2x_{d+1}}{\varepsilon}\Psi(T)\left(-\frac{x}{x_{d+1}} \frac{\varepsilon}{2}\right).$$
We iterate this procedure on all the dimensions till to \( n \), defining it well in \( Q^1 \cup C_2 \). Since \( \Psi_{|\partial(Q^1 \cup C_2)} = \text{Id} \) we can extend the map as the identity outside \( Q^1 \cup C_2 \). In addition one can easily check that \( \Psi \in D(x_0, r) \) and thus, since \( \Phi_{\varepsilon,E} \in D(x_0, r) \) and the class \( D(x_0, r) \) is closed by composition, this concludes the proof.

Later we will need to implement the above deformation of a set \( E \) on a rectangle rather than a cube: the deformation theorem can be proved for very general cubical complexes, \( \text{[Alm86]} \); however, for the sake of exposition, we limit ourselves to note the following simple observation: given a closed rectangle \( R := R_{x,a,b} \), using a linear map, and covering this time with rectangles homothetic to \( R \), one can easily draw the same conclusions as in Theorem 2.4. The only relevant difference is the area estimate (6), which holds with a constant \( k_1 \) depending on the ratio \( a/b \).

We will apply this construction to rectangles where the ratio is between 1 and 4.

### 3. Proof of Theorem 1.3

**Proof of Theorem 1.3** Up to extracting subsequences we can assume the existence of a Radon measure \( \mu \) on \( \mathbb{R}^n \setminus H \) such that

\[
\mu_j \rightharpoonup^* \mu, \quad \text{as Radon measures on } \mathbb{R}^n \setminus H,
\]

where \( \mu_j = \mathcal{H}^d \llcorner K_j \). We set \( K = \text{spt } \mu \setminus H \) and divide the argument in several steps.

**Step one.** We show the existence of \( \theta_0 = \theta_0(n,d) > 0 \) such that

\[
\mu(B_{x,r}) \geq \theta_0 \omega_d r^d, \quad x \in \text{spt } \mu \text{ and } r < d_x := \text{dist}(x,H)
\]

To this end it is sufficient to prove the existence of \( \beta = \beta(n,d) > 0 \) such that

\[
\mu(Q_{x,l}) \geq \beta l^d, \quad x \in \text{spt } \mu \text{ and } l < \frac{2d_x}{\sqrt{n}}.
\]

Let us assume by contradiction that there exist \( x \in \text{spt } \mu \) and \( l < \frac{2d_x}{\sqrt{n}} \) such that

\[
\frac{\mu(Q_{x,l})}{l} < \beta.
\]

We claim that this assumption, for \( \beta \) chosen sufficiently small depending only on \( d \) and \( n \), implies that for some \( l_\infty \in (0,l) \)

\[
\mu(Q_{x,l_\infty}) = 0,
\]

which is a contradiction with the property of \( x \) to be a point of \( \text{spt } \mu \). In order to prove (3.3), we assume that \( \mu(\partial Q_{x,l}) = 0 \), which is true for a.e. \( l \).

To prove (3.3), we construct a sequence of nested cubes \( Q_i = Q_{x,l_i} \) such that, if \( \beta \) is sufficiently small, it holds:

(i) \( Q_0 = Q_{x,l}; \)

(ii) \( \mu(\partial Q_{x,l_i}) = 0; \)

(iii) setting \( m_i := \mu(Q_i) \) it holds:

\[
\frac{m_i^{\frac{1}{2}}}{l_i} < \beta;
\]

(iv) \( m_{i+1} \leq (1 - \frac{1}{k_1})m_i \), where \( k_1 \) is the constant in Theorem 2.4 (6);

(v) \((1 - 4\varepsilon_i)l_i \geq l_{i+1} \geq (1 - 6\varepsilon_i)l_i \), where

\[
\varepsilon_i := \frac{1}{k\beta} \frac{m_i^{\frac{1}{2}}}{l_i}
\]

and \( k = \max \{6,6/(1 - (\frac{k_1 - 1}{k_1})^\frac{1}{2})\} \) is a universal constant.

(vi) \( \lim_i m_i = 0 \) and \( \lim_i l_i > 0 \).
Following [DS00], we are going to construct the sequence of cubes by induction: the cube $Q_0$ satisfies by construction hypotheses (i)-(iii). Suppose that cubes until step $i$ are already defined.

Setting $m_i^j := \mathcal{H}^d(K_j \cap Q_i)$ we cover $Q_i$ with the family $\Lambda_{\varepsilon_i}(Q_i)$ of closed cubes with edge length $\varepsilon_i l_i$ as described in Section 2 and we set $C_1^j$ and $C_2^j$ for the corresponding sets defined in (2.4). We define $Q_{i+1}$ to be the internal cube given by the construction, and we note that $C_1^j$ and $Q_{i+1}$ are non-empty if, for instance, $\varepsilon_i l_i$ is sufficiently small (\Phi 1/k \leq 1/6, which is guaranteed by our choice of $k$. Observe moreover that $C_1^j \cup C_2^j$ is a strip of width at most $2\varepsilon_i l_i$ around $\partial Q_i$, hence the side $l_{i+1}$ of $Q_{i+1}$ satisfies $(1 - 4\varepsilon_i)l_i \leq l_{i+1} < (1 - 2\varepsilon_i)l_i$.

Now we apply Theorem 2.4 to $Q_i$ with $E = K_j$ and $\varepsilon = \varepsilon_i l_i$, obtaining the map $\Phi_{i,j} = \Phi_{\varepsilon_i,l_i,K_j}$. We claim that, for every $j$ sufficiently large,

$$m_i^j \leq k_1(m_i^j - m_{i+1}^j) + o_j(1).$$

(3.5) Indeed, since $(K_j)$ is a minimizing sequence, by the definition of good class we have that

$$m_i^j \leq m_i + o_j(1) \leq \mathcal{H}^d(\Phi_{i,j}(K_j \cap Q_0)) + o_j(1)
= \mathcal{H}^d(\Phi_{i,j}(K_j \cap Q_{i+1})) + \mathcal{H}^d(\Phi_{i,j}(K_j \cap (C_1^j \cup C_2^j))) + o_j(1)
\leq k_1 \mathcal{H}^d(K_j \cap (C_1^j \cup C_2^j)) + o_j(1) = k_1(m_i^j - m_{i+1}^j) + o_j(1).$$

The last inequality holds because $\mathcal{H}^d(\Phi_{i,j}(K_j \cap Q_{i+1})) = 0$, otherwise by property (5) of Theorem 2.4 there would exists $T \in \Lambda^*_{\varepsilon_i,l_i}(Q_{i+1})$ such that $\mathcal{H}^d(\Phi_{i,j}(K_j \cap T)) = \mathcal{H}^d(T)$. Together with property (ii) this would imply

$$l_{i+1}^d = \mathcal{H}^d(T) \leq \mathcal{H}^d(\Phi_{i,j}(K_j \cap Q_i)) \leq k_1 \mathcal{H}^d(K_j \cap Q_i) \leq k_1 m_i^j \rightarrow k_1 m_i,$$

and therefore substituting (3.4)

$$m_i^j \leq k_1 m_i,$$

which is false if $\beta$ is sufficiently small ($m_i > 0$ because $x \in \text{spt}(\mu)$). Passing to the limit in $j$ in (3.5) we obtain (iv):

$$m_{i+1} \leq \frac{k_1 - 1}{k_1} m_i.$$  

(3.6)

Since $l_{i+1} \geq (1 - 4\varepsilon_i)l_i$ we can slightly shrink the cube $Q_{i+1}$ to a concentric cube $Q'_{i+1}$ with $l_{i+1}' \geq (1 - 6\varepsilon_i)l_i > 0$, $\mu(\partial Q'_{i+1}) = 0$ and for which (iv) still holds, since just $m_{i+1}$ decreased. With a slight abuse of notation we rename this last cube $Q'_{i+1}$ as $Q_{i+1}$.

We now show (iii). Using (3.6) and condition (iii) for $Q_i$ we obtain

$$\frac{m_{i+1}^j}{l_{i+1}^j} \leq \left( \frac{k_1 - 1}{k_1} \right)^\frac{1}{2} \frac{m_i^j}{(1 - 6\varepsilon_i)l_i} \leq \left( \frac{k_1 - 1}{k_1} \right)^\frac{1}{2} \frac{\beta}{1 - 6\varepsilon_i}.$$  

The last quantity will be less than $\beta$ if

$$\left( \frac{k_1 - 1}{k_1} \right)^\frac{1}{2} \leq 1 - 6\varepsilon_i = 1 - \frac{6 m_i^j}{k \beta l_i}.$$  

(3.7)

In turn inequality (3.7) is true because (iii) holds for $Q_i$, provided we choose $k \geq 6/(1 - (1 - 1/k_1)^{\frac{1}{2}})$. Furthermore, estimating $\varepsilon_0 < 1/k$ by (iii) and (v) we also have $\varepsilon_{i+1} \leq \varepsilon_i$. 


We are left to prove (vi): \( \lim_i m_i = 0 \) follows directly from (iv); regarding the non degeneracy of the cubes, note that

\[
\frac{l_\infty}{l_0} := \lim \inf_i \frac{l_i}{l_0} \geq \prod_{i=0}^{\infty} \left( 1 - 6 \varepsilon_i \right) = \prod_{i=0}^{\infty} \left( 1 - \frac{6 m_i^2}{k \beta_i} \right) \\
\geq \prod_{i=0}^{\infty} \left( 1 - \frac{6 \varepsilon_i}{k \beta_i} \prod_{i=0}^{\infty} \left( 1 - 6 \varepsilon \right) \right) \\
\geq \prod_{i=0}^{\infty} \left( 1 - \frac{6}{k \left( 1 - 6 \varepsilon \right)^{\frac{1}{k}}} \left( \frac{k_1 - 1}{k_1} \right)^{\frac{1}{k}} \right) ,
\]

where we used \( \varepsilon \leq \varepsilon_0 \) in the last inequality. Since \( \varepsilon_0 < 1/k \) the last product is strictly positive provided

\[
k > \frac{6}{1 - \left( \frac{k_1 - 1}{k_1} \right)^{\frac{1}{k}}},
\]

which is guaranteed by our choice of \( k \).

**Step two:** We fix \( x \in \text{spt} \mu \setminus H \), and prove that

\[
r \mapsto \frac{\mu(B_{x,r})}{r^d} \quad \text{is increasing on } (0, d_x).
\]

The proof is a straightforward adaptation of the corresponding one in [DGM14, Theorem 2], and amounts to prove a differential inequality for the function \( f(r) := \mu(B_{x,r}) \). In turn, this inequality is obtained in a two step approximation: first one exploits the rectifiability of the minimizing sequence \( (K_j) \) and property (1.1) to compare \( K_j \) with the cone competitor \( C_{x,r}(K_j) \), see (2.1). The comparison a priori is only allowed with elements of \( P(H) \), so for almost every \( r < d_x \) it holds:

\[
f_j(r) = \mathcal{H}^d(K_j) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \leq m_0 + o_j(1) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \leq o_j(1) + \inf_{K' \in P(H) \setminus B_{x,r} \setminus B_{x,r}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}),
\]

where \( f_j(r) := \mathcal{H}^d(K_j \cap \overline{B_{x,r}}) \). Nevertheless \( K_j \) can be compared with its cone competitor, up to an error infinitesimal in \( j \), thanks to Lemma (2.3). We recover

\[
\inf_{K' \in P(H) \setminus B_{x,r} \setminus B_{x,r}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}) \leq o_j(1) + \frac{r}{d} \mathcal{H}^d(K_j \cap \partial B_{x,r}) = o_j(1) + \frac{r}{d} f_j'(r). \]

One then passes to the limit in \( j \) and obtains the desired monotonicity formula. We refer to [DGM14, Theorem 2] for the conclusion of the proof of (3.8).

**Step three:** By (3.2) and (3.8) the \( d \)-dimensional density of the measure \( \mu \), namely:

\[
\theta(x) = \lim_{r \to 0^+} \frac{f(r)}{\omega_d r^d} \geq \theta_0,
\]

exists, is finite and positive \( \mu \)-almost everywhere. Preiss’ Theorem (2.1) implies that \( \mu = \theta \mathcal{H}^d \ll \tilde{K} \) for some countably \( \mathcal{H}^d \)-rectifiable set \( \tilde{K} \) and some positive Borel function \( \theta \). Since \( K \) is the support of \( \mu \), \( \mathcal{H}^d(\tilde{K} \setminus K) = 0 \). On the other hand, by differentiation of Hausdorff measures, (3.2) yields \( \mathcal{H}^d(K \setminus \tilde{K}) = 0 \). Hence \( K \) is \( d \)-rectifiable and \( \mu = \theta \mathcal{H}^d \ll K \).
Step four: We prove that \( \theta(x) \geq 1 \) for every \( x \in K \) such that the approximate tangent space to \( K \) exists (thus, \( \mathcal{H}^d \)-a.e. on \( K \)). Fix any such \( x \in K \setminus H \) without loss of generality we can suppose that \( x = 0 \) and that \( \pi = \{ x_{d+1} = \ldots = x_n = 0 \} \) is the approximate tangent space to \( K \) at 0: in particular,
\[
\mathcal{H}^d \bigg\{ \frac{K}{r} \bigg\} \rightarrow^* \mathcal{H}^d \bigg\{ \pi \bigg\}, \quad \text{as } r \to 0^+.
\]
The above convergence, together with the lower density estimates \((3.2)\) imply that, for every \( \varepsilon > 0 \), there is \( \rho > 0 \) such that
\[
K \cap B_r \subset \left\{ y \in \mathbb{R}^n : |y_{d+1}|, \ldots, |y_n| < \frac{\varepsilon}{2} r \right\} \quad \forall r < \rho. \tag{3.9}
\]

Let us now assume, by contradiction, that \( \theta(0) < 1 \). Thanks to \((3.8)\) and \((3.9)\), there exist \( r \in (0, d_\varepsilon) \) and \( \alpha < 1 \) such that \( \mu(\partial B_r) = 0 \) and
\[
\frac{\mu(Q_\rho)}{\rho^d} \leq \alpha < 1, \quad K \cap (Q_\rho \setminus R_{\rho, \varepsilon \rho}) = \emptyset \quad \forall \rho \leq r. \tag{3.10}
\]

In particular, since \( \mu_j \) are weakly converging to \( \mu \) we get that for \( j \) large
\[
\frac{\mu_j(Q_r)}{r^d} \leq \alpha < 1 \quad \text{and} \quad \mu_j(Q_r \setminus R_{r, \varepsilon r}) = o_j(1), \tag{3.11}
\]
We now wish to clear the small amount of mass appearing in the complement of \( R_{r, \varepsilon r} \): we achieve this by repeatedly applying Theorem \(2.4\). We set \( Q_r \cap \{ x_{d+1} \geq \frac{\varepsilon}{2} r \} =: R \) and we apply Theorem \(2.4\) to this rectangle with parameter \( \varepsilon \) and \( E = K^0_j := K_j \), obtaining the map \( \varphi_{1,j} \). We recall that the obtained constant \( k_1 \) for the area bound is universal, since it depends on the side ratio of \( R \), which is bounded from below by 1 and from above by \( 4 \), provided \( \varepsilon \) small enough. We set \( K^1_j := \varphi_{1,j}(K^0_j) \) and repeat the argument with \( Q_r \cap \{ x_{d+1} \leq -\frac{\varepsilon}{2} r \} =: R \) and \( E := K^1_j \), obtaining the map \( \varphi_{2,j} \). We again set \( K^2_j := \varphi_{2,j}(K^1_j) \) and iterate this procedure to the rectangles \( Q_r \cap \{ x_{d+2} \geq \frac{\varepsilon}{2} r \}, \ldots, Q_r \cap \{ x_n \leq -\frac{\varepsilon}{2} r \} \). After \( 2(n-d) \) iteration, we set
\[
K^{2(n-d)}_j := \varphi_{2(n-d),j} \circ \cdots \circ \varphi_{1,j}(K_j).
\]
We are going to use the cube \( Q_{r(1-\sqrt{\varepsilon})} \) because, taking \( \varepsilon \) small enough, then \( \sqrt{\varepsilon} > 4C\varepsilon \), where \( C > 1 \) is the side ratio considered before. This allows us to claim that
\[
\mathcal{H}^d(K^{2(n-d)}_j \cap (Q_{r(1-\sqrt{\varepsilon})} \setminus R_{r(1-\sqrt{\varepsilon}),6\varepsilon r})) = 0. \tag{3.12}
\]
Otherwise there would exist a \( d \)-face of a smaller rectangle \( T \subset (Q_r \setminus R_{r, \varepsilon r}) \) such that
\[
\mathcal{H}^d(K^{2(n-d)}_j \cap T) = \mathcal{H}^d(T) \geq \varepsilon^d r^d,
\]
which would lead to the following contradiction for \( j \) large:
\[
\varepsilon^d r^d \leq \mathcal{H}^d(T) \leq \mathcal{H}^d(K^{2(n-d)}_j \cap (Q_r \setminus R_{r, \varepsilon r})) \leq k_1^{2(n-d)} \mathcal{H}^d(K_j \cap (Q_r \setminus R_{r, \varepsilon r})) = o_j(1)
\]
In particular, we cleared any measure on every slab
\[
\bigcup_{i=d+1}^n \left\{ 3\varepsilon r < |x_i| < (1 - \sqrt{\varepsilon})^2 \right\} \cap Q_{r(1-\sqrt{\varepsilon})}.
\]
We want now to construct a map \( P \in D(0, r) \), collapsing \( R_{r(1-\sqrt{\varepsilon}),6\varepsilon r} \) onto the tangent plane. To this end, for \( x \in \mathbb{R}^n \), \( x = (x', x'') \) with \( x' \in \mathbb{R}^d \) and \( x'' \in \mathbb{R}^{n-d} \) we set
\[
\|x'\| := \max\{|x_i| : i = 1, \ldots, d\} \quad \|x''\| := \max\{|x_i| : i = d+1, \ldots, n\} \tag{3.13}
\]
and we define $P$ as follows:

$$P(x) = \begin{cases} (x', g(\|x'\|) \frac{\|x''\| - 3\|x''\|}{1 - \beta} + \frac{x''}{\|x''\|} + (1 - g(\|x'\|))x'') & \text{if } \max\{\|x'\|, \|x''\|\} \leq r/2 \\ \operatorname{Id} & \text{otherwise,} \end{cases}$$  \tag{3.14}

where $g : [0, r/2] \to [0, 1]$ is a compactly supported cut off function such that

$$g \equiv 1 \text{ on } [0, r(1 - \sqrt{\varepsilon})/2] \quad \text{and} \quad |g'| \leq 10/r \sqrt{\varepsilon}.$$  

It is not difficult to check that $P \in \mathcal{D}(0, r)$ and that $\operatorname{Lip} P \leq 1 + C \sqrt{\varepsilon}$ for some dimensional constant $C$.

We now set $\widetilde{K}_j := P(K_j^{2(n-d)})$, which verifies, thanks to (3.12),

$$\mathcal{H}^d\left(\widetilde{K}_j \cap (Q_{(1-\sqrt{\varepsilon})r} \setminus Q^d_{(1-\sqrt{\varepsilon})r})\right) = 0 \tag{3.15}$$

and

$$\mathcal{H}^d\left(\widetilde{K}_j \cap (Q_r \setminus Q_{r(1-\sqrt{\varepsilon})})\right) \leq (1 + C \sqrt{\varepsilon})^d \mathcal{H}^d(K_j^{2(n-d)} \cap (Q_r \setminus Q_{r(1-\sqrt{\varepsilon})}))$$

$$\leq (1 + C \sqrt{\varepsilon})^d K_j^{2(n-d)} \mathcal{H}^d(K_j \cap (Q_r \setminus (Q_{r(1-\sqrt{\varepsilon})} \cup R_{r,cr})))$$

$$+ (1 + C \sqrt{\varepsilon}) \mathcal{H}^d(K_j \cap (R_{r,cr} \setminus Q_{r(1-\sqrt{\varepsilon})}))$$

$$\leq o_j(1) + (1 + C \sqrt{\varepsilon}) \mathcal{H}^d(K_j \cap (R_{r,cr} \setminus Q_{r(1-\sqrt{\varepsilon})})),$$

where in the last inequality we have used (3.11). Moreover, by using (3.10), (3.11) and (3.15) we also have that, for $\varepsilon$ small and $j$ large:

$$\frac{\mathcal{H}^d(\widetilde{K}_j \cap Q^d_{r(1-\sqrt{\varepsilon})})}{r^d(1 - \sqrt{\varepsilon})^d} \leq \frac{\mathcal{H}^d(\widetilde{K}_j \cap Q^d_{r(1-\sqrt{\varepsilon})})}{r^d(1 - \sqrt{\varepsilon})^d} \leq (1 + C \sqrt{\varepsilon}) \frac{\mathcal{H}^d(K_j^{2(n-d)} \cap Q_r)}{r^d}$$

$$\leq (1 + C \sqrt{\varepsilon}) \frac{\mathcal{H}^d(K_j \cap Q_r) + o_j(1)}{r^d} \leq \alpha + o_j(1) < 1. \tag{3.17}$$

As a consequence of (3.17) and the compactness of $\widetilde{K}_j$, there exist $y'_j \in Q^d_{(1-\sqrt{\varepsilon})r}$ and $\delta_j > 0$ such that, if we set $y_j := (y'_j, 0)$, then

$$\widetilde{K}_j \cap B_{y_j, \delta_j} = \emptyset \quad \text{and} \quad B_{y_j, \delta_j} \subset Q^d_{(1-\sqrt{\varepsilon})r}. \tag{3.18}$$

After the last deformation, our set $\widetilde{K}_j \cap Q_{r(1-\sqrt{\varepsilon})}$ lives on the tangent plane and we want to use the property (3.18) to collapse $\widetilde{K}_j \cap Q_{r(1-\sqrt{\varepsilon})}$ into $\left(\partial Q^d_{(1-\sqrt{\varepsilon})r}\right) \times \{0\}^{n-d}$. To this end let us define for every $j \in \mathbb{N}$, the following Lipschitz map:

$$\varphi_j(x) = \begin{cases} (x' + z'_{j,x}, x'') & \text{if } x \in R_{r(1-\sqrt{\varepsilon})r} \\ x & \text{otherwise,} \end{cases}$$

with

$$z'_{j,x} := \min \left\{ 1, \frac{|x' - y'_j|}{\delta_j} \right\} (r - 4\|x''\|) \frac{\gamma_{j,x}(x' - y'_j)}{r},$$

where $\gamma_{j,x} > 0$ is such that $x' + \gamma_{j,x}(x' - y'_j) \in \partial Q^d_{(1-\sqrt{\varepsilon})r} \times \{0\}^{n-d}$ and $\|x''\|$ is defined in (3.13). One can easily check that $\varphi_j \in \mathcal{D}(0, r)$. Moreover, setting $\varphi_j(\widetilde{K}_j) =: K_j'$ we have that

$$K_j' \setminus Q_r = K_j \setminus Q_r$$

and

$$\mathcal{H}^d(K_j' \cap Q_{r(1-\sqrt{\varepsilon})}) = 0. \tag{3.19}$$
thanks to (3.15), since
\[ \mathcal{H}^d \left( \partial Q_r^{(1-\sqrt{\varepsilon})} \times \{ 0 \}^{n-d} \right) = 0. \]
Since \( \mathcal{P}(H) \) is a good class, by (1.1) there exists a sequence of competitors \( (J_j)_{j \in \mathbb{N}} \subset \mathcal{P}(H) \) such that \( J_j \setminus B_{0,r} = K_j \setminus B_{0,r} \) and \( \mathcal{H}^d(J_j) = \mathcal{H}^d(K_j') + o_j(1) \). Hence, thanks to (3.16) and (3.19) we get
\[
\mathcal{H}^d(K_j) - \mathcal{H}^d(J_j) \geq \mathcal{H}^d(K_j) - \mathcal{H}^d(K_j') - o_j(1) = \mathcal{H}^d(K_j \cap Q_r) - \mathcal{H}^d(K_j \cap Q_r) - o_j(1)
\[
\geq \mathcal{H}^d \left( K_j \cap Q_r(1-\sqrt{\varepsilon}) \right) + \mathcal{H}^d \left( K_j \cap (R_{r,\varepsilon} \setminus Q_r(1-\sqrt{\varepsilon})) \right)
\[
- o_j(1) - (1 + C \sqrt{\varepsilon}) \mathcal{H}^d \left( K_j \cap (R_{r,\varepsilon} \setminus Q_r(1-\sqrt{\varepsilon})) \right)
\[
\geq \mathcal{H}^d \left( K_j \cap Q_r(1-\sqrt{\varepsilon}) \right) - C \sqrt{\varepsilon} \mathcal{H}^d \left( K_j \cap (R_{r,\varepsilon} \setminus Q_r(1-\sqrt{\varepsilon})) \right) - o_j(1).
\]
Passing to the limit as \( j \to \infty \), and using (3.1), (3.2) and (3.10) we get that
\[
\liminf_j \mathcal{H}^d(K_j) \geq \liminf_j \mathcal{H}^d(J_j) + \mu(Q_r(1-\sqrt{\varepsilon})) - C \sqrt{\varepsilon} r^d
\[
\geq \liminf_j \mathcal{H}^d(J_j) + (\theta_0(1 - \sqrt{\varepsilon})^d - C \sqrt{\varepsilon}) r^d.
\]
Since, for \( \varepsilon \) small, this is in contradiction with \( K_j \) be a minimizing sequence we finally conclude that \( \theta(0) \geq 1 \).

**Step five.** We prove that \( \theta(x) \leq 1 \) for every \( x \in K \) such that the approximate tangent space to \( K \) exists. Arguing by contradiction, we assume that \( \theta(x) = 1 + \sigma > 1 \) for some \( x \) where \( K \) admits an approximate tangent plane \( T \). As usually we assume that \( x = 0 \) and \( T = \{ y : y_{d+1}, \ldots, y_n = 0 \} \).

By the monotonicity of the density established in Step 2, for every \( \varepsilon > 0 \) we can find \( r_0 > 0 \) such that
\[
K \cap Q_r \subset R_{r,\varepsilon}, \quad 1 + \sigma \leq \frac{\mu(Q_r)}{r^d} \leq 1 + \sigma + \varepsilon \sigma, \quad \forall r \leq r_0.
\]
Let us also note that for every \( r < r_0 \) there exists \( j_0 \) such that
\[
\mathcal{H}^d(K_j \cap Q_r) > \left( 1 + \frac{\sigma}{2} \right) r^d, \quad \mathcal{H}^d((K_j \cap Q_r) \setminus R_{r,\varepsilon}) < \frac{\sigma}{4} r^d, \quad \forall r \geq j_0.
\]
Consider the map \( P : \mathbb{R}^n \to \mathbb{R}^n \in \text{D}(0,r) \) with \( \text{Lip} \ P \leq 1 + C \sqrt{\varepsilon} \) defined in (3.14) which collapses \( R_{r(1-\sqrt{\varepsilon}),\varepsilon} \) onto the tangent plane. By exploiting the fact the \( \mathcal{P}(H) \) is a good class we find that
\[
\mathcal{H}^d(K_j \cap Q_r) - o_j(1) \leq \mathcal{H}^d(P(K_j \cap R_{r(1-\sqrt{\varepsilon}),\varepsilon})) + \mathcal{H}^d(P(K_j \cap (R_{r,\varepsilon} \setminus R_{r(1-\sqrt{\varepsilon)},\varepsilon})))
\]
\[
+ \mathcal{H}^d(P(K_j \cap (Q_r \setminus R_{r,\varepsilon}))).
\]
By construction, \( I_1 \leq r^d \), while, by (3.21), \( \mathcal{H}^d(K_j \cap Q_r) > (1 + (\sigma/2)) r^d \) and
\[
I_3 \leq (\text{Lip} \ P)^d \mathcal{H}^d(K_j \cap (Q_r \setminus R_{r,\varepsilon})) < (1 + C \sqrt{\varepsilon})^d \frac{\sigma}{4} r^d.
\]
Hence, as \( j \to \infty \),
\[
\left( 1 + \frac{\sigma}{2} \right) r^d \leq r^d + \liminf_{j \to \infty} I_2 + (1 + C \sqrt{\varepsilon})^d \frac{\sigma}{4} r^d,
\]
that is,
\[
\left( \frac{1}{2} - \frac{(1 + C \sqrt{\varepsilon})^d}{4} \right) \sigma \leq \liminf_{j \to \infty} \frac{I_2}{r^d}.
\]
By (3.20), we finally estimate that
\[
\limsup_{j \to \infty} I_2 \leq (1 + C\sqrt{\varepsilon})^d \mu(Q_r \setminus Q_{(1-\sqrt{\varepsilon})r})
\leq (1 + C\sqrt{\varepsilon})^d \left( (1 + \sigma + \varepsilon\sigma) - (1 + \sigma)(1 - \sqrt{\varepsilon})^d \right) r^d
\]  
(3.23)
By choosing \( \varepsilon \) sufficiently small, (3.22) and (3.23) provide the desired contradiction. Hence \( \theta \leq 1 \) and, combining this with the previous step, \( \mu = \mathcal{H}^d \mathcal{L} K \).

**Step six:** We now show that the canonical density one rectifiable varifold associated to \( K \) is stationary in \( \mathbb{R}^n \setminus H \). In particular, applying Allard’s regularity theorem, see [Sim83, Chapter 5], we will deduce that there exists an \( \mathcal{H}^d \)-negligible closed set \( \Sigma \subset K \) such that \( \Gamma = K \setminus \Sigma \) is a real analytic manifold. Since being a stationary varifold is a local property, to prove our claim it is enough to show that for every ball \( B \subset \subset \mathbb{R}^n \setminus H \) we have
\[
\mathcal{H}^d(K) \leq \mathcal{H}^d(\phi(K))
\]  
(3.24)
whenever \( \phi \) is a diffeomorphism such that \( \text{spt}(\phi - \text{Id}) \subset B \). Indeed, by exploiting (3.24) with \( \phi_t = \text{Id} + t\mathbf{x}, \mathbf{x} \in C^1(B) \) we deduce the desired stationarity property.

To prove (3.24) we argue as in [DGM14, Theorem 7]. Given \( \varepsilon > 0 \) we can find \( \delta > 0 \) and a compact set \( K \subset K \cap B \) with \( \mathcal{H}^d((K \setminus K) \cap B) < \varepsilon \) such that \( K \) admits an approximate tangent plane \( \pi(x) \) at every \( x \in K \),
\[
\sup_{x \in K} \sup_{y \in K} |\nabla \phi(x) - \nabla \phi(y)| \leq \varepsilon, \quad \sup_{x \in K} \sup_{y \in K \setminus B} d(\pi(x), \pi(y)) < \varepsilon,
\]  
(3.25)
where \( d \) is a distance on \( G(d) \), the \( d \)-dimensional Grassmanian. Moreover, denoting by \( S_{x,r} \) the set of points in \( B_{x,r} \) at distance at most \( \varepsilon \) from \( x + \pi(x) \), then \( K \cap B_{x,r} \subset S_{x,r} \) for every \( r < \delta \) and \( x \in K \). By Besicovitch covering theorem we can find a finite disjoint family of closed balls \( \{B_i\} \) with \( B_i = B_{x_i,r_i} \subset B \subset \subset \mathbb{R}^n \setminus H \), \( x_i \in K \), and \( r_i < \delta \), such that \( \mathcal{H}^d(K \setminus \bigcup B_i) < \varepsilon \). By exploiting the construction of Step four, we can find \( j(\varepsilon) \in \mathbb{N} \) and maps \( P_i : \mathbb{R}^n \to \mathbb{R}^n \) with \( \text{Lip}(P_i) \leq 1 + C\sqrt{\varepsilon} \) and \( P_i = \text{Id} \) on \( B_i \), such that, for a certain \( X_i \subset S_i = S_{x_i, \varepsilon r_i} \),
\[
P_i(X_i) \subset B_i \cap (x_i + \pi(x_i)),
\]  
(3.26)
\[
\mathcal{H}^d(P_i((K \setminus K) \cap B_i) \setminus X_i) \leq C\sqrt{\varepsilon} \omega_d r_i^d, \quad \forall j \geq j(\varepsilon).
\]  
Denoting with \( J_d^\pi \) the \( d \)-dimensional tangential jacobian with respect to the plane \( \pi \) and by \( J_d^K \) the one with respect to \( K \) and exploiting (3.25), (3.26), the area formula and that \( \omega_d r_i^d \leq \mathcal{H}^d(K \cap B_i) \) (by the monotonicity formula), and setting \( \alpha_i = \mathcal{H}^d((K \setminus K) \cap B_i) \), we get
\[
\mathcal{H}^d(\phi(P_i(K_j \cap X_i))) = \mathcal{H}^d(\phi(P_i(K_j \cap X_i))) = \int_{P_i(K_j \cap X_i)} J_d^{\pi(x_i)}(x) d\mathcal{H}^d(x) \leq (J_d^{\pi(x_i)}(x_i + \varepsilon) + \varepsilon) \omega_d r_i^d
\leq (J_d^{\pi(x_i)}(x_i) + \varepsilon) \mathcal{H}^d(K \cap B_i) \leq (J_d^{\pi(x_i)}(x_i) + \varepsilon) (\mathcal{H}^d(K \cap B_i) + \alpha_i)
\leq \int_{K \cap B_i} (J_d^K(\phi + \varepsilon) d\mathcal{H}^d(x) + ((\text{Lip}(\phi))^d + \varepsilon) \alpha_i
= \mathcal{H}^d(\phi(K \cap B_i)) + 2\varepsilon \mathcal{H}^d(K \cap B_i) + ((\text{Lip}(\phi))^d + \varepsilon) \alpha_i,
\]  
(3.27)
where in the last identity we have used the area formula and the injectivity of \( \phi \). Since \( P_i = \text{Id} \) on \( B_i^c \), \( \phi = \text{Id} \) on \( B^c \), \( B_i \subset B \) and the balls \( B_i \) are disjoint, the map \( \phi \) which is equal to \( \phi \) on \( B \setminus \bigcup B_i \), equal to the identity on \( B^c \) and equal to \( \phi \circ P_i \) on \( B_i \) is well defined. Moreover, by (3.27), we get
\[
\mathcal{H}^d(\phi(K_j)) \leq \mathcal{H}^d(\phi(K)) + C\varepsilon
\]
where \( C \) depends only on \( K \). By exploiting the definition of good class, we get that
\[
\mathcal{H}^d(K) \leq \mathcal{H}^d(\tilde{\phi}(K_j)) + o_j(1) \leq \mathcal{H}^d(\phi(K)) + C\varepsilon + o_j(1).
\]
Letting \( j \to \infty \) and \( \varepsilon \to 0 \) we obtain (3.24).

**Step seven:** We finally address the dimension of the singular set. Recall that, by monotonicity, the density function
\[
\Theta^d(K, x) = \lim_{r \to 0} \frac{\mathcal{H}^d(K \cap B_{x,r})}{\omega_{d}r^d}
\]
is everywhere defined in \( \mathbb{R}^n \setminus H \) and equals 1 \( \mathcal{H}^d \)-almost everywhere in \( K \). Fixing \( x \in K \) and a sequence \( r_k \downarrow 0 \), the monotonicity formula, the stationarity of \( \mathcal{H}^d \cap K \) and the compactness theorem for integral varifolds [All72, Theorem 6.4] imply that (up to subsequences)
\[
\mathcal{H}^d(K \setminus \frac{K - x}{r_k}) \to V \quad \text{locally in the sense of varifolds, (3.28)}
\]
with
(a) \( V \) is a stationary integral varifold: in particular \( \Theta^d(\|V\|, y) \geq 1 \) for \( y \in \text{spt}(V) \);
(b) \( V \) is a cone, namely \( \delta(\lambda x) \# V = V \), where \( \delta(x) = \lambda x, \lambda > 0 \);
(c) \( \Theta^d(\|V\|, 0) = \Theta^d(K, x) = \Theta^d(\|V\|, y) \) for every \( y \in \mathbb{R}^n \).

Recall that the tangent varifold \( V \) depends (in principle) on the sequence \((r_k)\). We denote by \( \text{TanVar}(K, x) \) the (nonempty) set of all possible limits \( V \) as in (3.28) varying among all sequences along which (3.28) holds. Given a cone \( W \) we set
\[
\text{Spine}(W) := \{ y \in \mathbb{R}^n : \Theta^d(\|W\|, y) = \Theta^d(\|W\|, 0) \} : \quad (3.29)
\]
by [Alm00] 2.26 \( \text{Spine}(W) \) is a vector subspace of \( \mathbb{R}^n \), see also [Whi97] Theorem 3.1. We can stratify \( K \) in the following way: for every \( k = 0, \ldots, n \) we let
\[
A_k := \{ x \in K : \text{ for all } V \in \text{TanVar}(K, x), \dim \text{Spine}(V) \leq k \}.
\]
Clearly \( A_0 \supseteq \cdots \supseteq A_{d+1} = \cdots = A_n = \emptyset \); moreover it holds: \( \dim_{\mathcal{H}} A_k \leq k \), see [Alm00, 2.28] and [Whi97] Theorem 2.2. In order to prove our claim, we need to show that \( A_d \setminus A_{d-1} \subset K \setminus \Sigma \), where \( \Sigma \) as in Step six is the singular set of \( K \), namely that every point in \( K \) having at least one tangent cone of maximal dimension \( d \) must be regular (note that, as the example of complex varieties shows, this is not true in general for a stationary varifold).

First, note that if \( x \in A_d \setminus A_{d-1} \) and \( V \in \text{TanVar}(K, x) \) satisfies \( \dim \text{Spine}(V) = d \), then \( V = \mathcal{H}^d \cap \text{Spine}(V) \): indeed up to a rotation \( \text{spt}(V) = \text{Spine}(V) \times \Gamma \), where \( \Gamma \) is a cone in \( \mathbb{R}^{n-d} \). If \( \Gamma \neq \{0\} \) then \( \Theta^d(\|V\|, 0) > \Theta^d(\|V\|, y) \) for any \( y \in \text{Spine}(V) \setminus \{0\} \), which contradicts (3.29). Hence by (3.29) and (c), \( \Theta^d(\|V\|, 0) \) is an integer and \( V = \Theta^d(K, x) \mathcal{H}^d \cap \text{Spine}(V) \). Second, the density lower bound \( \Theta^d(K, \cdot) \geq 1 \) and (3.28) imply that for every \( \varepsilon > 0 \)
\[
\frac{K - x}{r} \subset U_\varepsilon(\text{Spine}(V)) \quad \text{and} \quad \mathcal{H}^d \left( \frac{K - x}{r} \cap Q_{0,1} \right) \leq (1 + \varepsilon) \Theta^d(K, x)
\]
if \( r \) is sufficiently small. By arguing as in Step five (i.e. roughly speaking comparing \( K \) with \( P(K) \) in \( Q_{x,r} \), where \( P \) is the squeezing map (3.14), although one has to rigorously get through the minimizing sequence \( K_j \)) we obtain
\[
\mathcal{H}^d \left( \frac{K - x}{r} \cap Q_{0,1} \right) \leq (1 + C\sqrt{\varepsilon}).
\]
Letting \( r \downarrow 0 \) thanks to (3.28) we obtain \( \|V\|(Q_{0,1}) \leq (1 + C\sqrt{\varepsilon}) \), implying \( \Theta^d(K, x) = 1 \). We therefore fall into the hypotheses of Allard’s regularity Theorem [All72, Regularity Theorem, Section 8], \( K \cap Q_{x,\frac{x}{2}} \) is a real analytic submanifold. Equivalently \( x \notin \Sigma \). \( \square \)
4. Proof of Theorems 1.5 and 1.8

In this Section we prove Theorem 1.5 and 1.8. With Theorem 1.3 at hand, the proofs are quite similar to the corresponding ones in [DGM14] (see Theorems 4 and 7 there), hence we limit ourselves to provide a short sketch underlying only the main differences.

Proof of Theorem 1.5. We start by proving that $F(H,C)$ is a good class in the sense of Definition 1.2: let $\tilde{K} \in F(H,C)$, $x \in K$, $r \in (0, \text{dist}(x,H))$ and $\varphi \in D(x,r)$. We show that $\varphi(\tilde{K}) \in F(H,C)$ arguing by contradiction: assume that $\gamma(S^{n-d}) \cap \varphi(\tilde{K}) = \emptyset$ for some $\gamma \in C$ and, without loss of generality, suppose also that $\gamma(S^{n-d}) \cap (\tilde{K} \setminus B_{x,r}) = \emptyset$. By Definition 1.1 there exists a sequence $(\varphi_j) \subset D(x,r)$ such that $\lim_j \|\varphi_j - \varphi\|_{C^0} = 0$.

Since $\gamma(S^{n-d})$ is compact and $\varphi_j = \text{Id}$ outside $B_{x,r}$, for $j$ sufficiently large $\gamma(S^{n-d}) \cap \varphi_j(\tilde{K}) = \emptyset$; moreover $\varphi_j^{-1}(\gamma(S^{n-d})) \cap \tilde{K} = \emptyset$. But the property for $\varphi_j$ of being isotopic to the identity implies $\varphi_j^{-1} \circ \gamma \in C$, which contradicts $\tilde{K} \in F(H,C)$. This proves (a).

Given a minimizing sequence $(K_j) \subset F(H,C)$ which consists of rectifiable sets, we can therefore find a set $K$ with the properties stated in Theorem 1.3. In order to conclude (b), namely that $K \in F(H,C)$, we refer to [DGM14, Theorem 4(b)]: the proof is the same.

Proof of Theorem 1.8. As already observed in Remark 1.7, $A(H,K_0)$ is a good class and we can therefore apply Theorem 1.3. We thus know that $H^d \llcorner K_j \rightharpoonup \mu = H^d \llcorner K$ and that $K$ is a smooth set away from $H$ and from a relatively closed set $\Sigma$ of dimension less or equal than $(d - 1)$. The conclusion of the proof can now be obtained by repeating verbatim Steps 4 and 6 in the proof of Theorem 7 in [DGM14].

Bibliography

[DS00] G. David and S. Semmes. Uniform rectifiability and quasiminimizing sets of arbitrary 

[De54] E. De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad 

[De55] E. De Giorgi. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad 

[DeL08] C. De Lellis. *Rectifiable sets, densities and tangent measures*. Zurich Lectures in Ad-


1960.

2014. ISSN 1050-6926.

the 1976 original.


[Rei60] E. R. Reifenberg. Solution of the Plateau problem for $m$-dimensional surfaces of varying 

for Mathematical Analysis*. Australian National University, Centre for Mathematical 

[Whi97] Brian White. Stratification of minimal surfaces, mean curvature flows, and harmonic 

**INSTITUT FÜR MATHematik, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH, 
SWITZERLAND**

*E-mail address: guido.dephilippis@math.uzh.ch*

**INSTITUT FÜR MATHematik, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH, 
SWITZERLAND**

*E-mail address: antonio.derosa@math.uzh.ch*

**INSTITUT FÜR MATHematik, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH, 
SWITZERLAND**

*E-mail address: francesco.ghiraldin@math.uzh.ch*